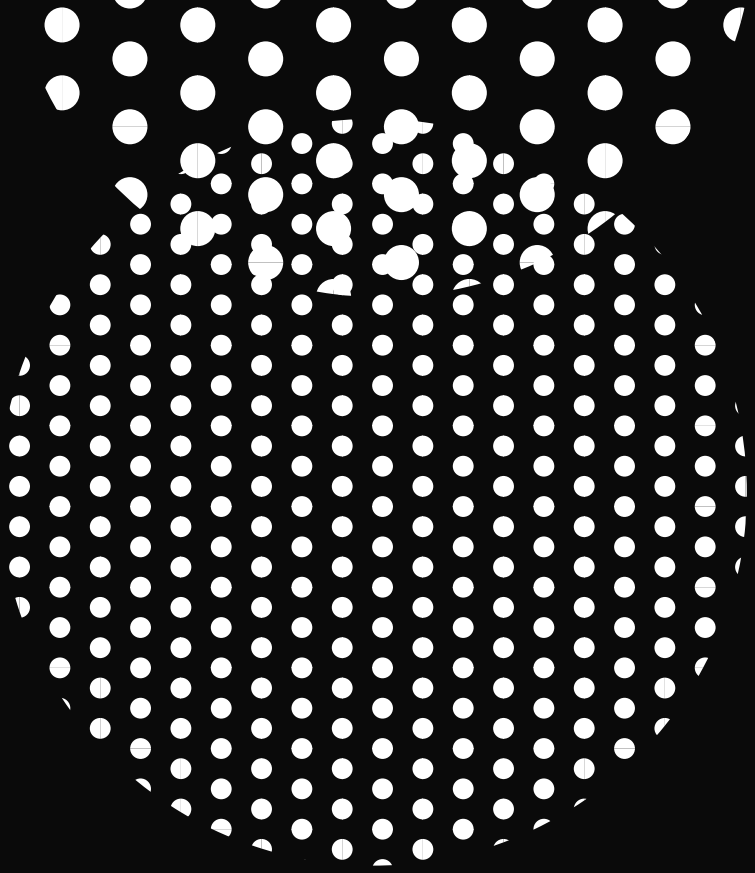

Exercice 27

a) $\dfrac{X^5+1}{X^2(X-1)^2} = P(X)$

$= aX + b + \dfrac{c}{X^2} + \dfrac{d}{X} + \dfrac{c'}{(X-1)^2} + \dfrac{d'}{X-1}$

1ère technique

$$\begin{array}{r|l} X^5+1 & X^4-2X^3+X^2 \\ \hline X^5-2X^4+X^3 & X+2 \\ \hline 2X^4-X^3+1 & \\ \hline 2X^4-4X^3+2X^2 & \\ \hline 3X^3-2X^2+1 & \end{array}$$

Alors $P(X) = X+2+\dfrac{3X^3-2X^2+1}{X^2(X-1)^2}$

2ème technique

$\dfrac{P(X)}{X} = \dfrac{X^5\left(1+\frac{1}{X^5}\right)}{X^5\left(1-\frac{1}{X}\right)^2} \xrightarrow[X\to+\infty]{} 1$

et $\dfrac{P(X)}{X} = a + \dfrac{b}{X} + \dfrac{c}{X^3} + \dfrac{d}{X^2} + \dfrac{c'}{X(X-1)^2} + \dfrac{d'}{X(X-1)} \xrightarrow[X\to+\infty]{} a$

donc $\underline{a=1}$

$P(X) - X = b + \dfrac{c}{X^2} + \dfrac{d}{X} + \dfrac{c'}{(X-1)^2} + \dfrac{d'}{(X-1)} \xrightarrow[X\to+\infty]{} b$

$P(X) - X = \dfrac{X^5+1}{X^2(X-1)} - X = \dfrac{X^5+1-(X^5-2X^4+X^3)}{X^2(X-1)^2} = \dfrac{2X^4-X^3+1}{X^2(X-1)^2} \xrightarrow[X\to+\infty]{} 2$

donc $\underline{b=2}$

$Q(X) = P(X) - (X+2) = \dfrac{2X^4-X^3+1}{X^2(X-1)^2} - 2$

$= \dfrac{2X^4-X^3+1-(2X^4-4X^3+2X^2)}{X^2(X-1)^2}$

$= \dfrac{3X^3-2X^2+1}{X^2(X-1)^2}$

$$X^2Q(X) = c + dX + \frac{c'X^2}{(X-1)^2} + \frac{d'X^2}{(X-1)^2} \xrightarrow[X\to 0]{} c$$

$$X^2Q(X) = \frac{3X^3-2X^2+1}{(X-1)^2} \xrightarrow[X\to 0]{} 1$$

donc $c = 1$

d'où $Q(X) - \frac{1}{X^2} = \frac{3X^3-2X^2+1-(X^2-2X+1)}{X^2(X-1)^2} = \frac{3X^3-3X^2+2X}{X^2(X-1)^2}$

$$= \frac{3X^2-3X+2}{X(X-1)^2}$$

$$\left(Q(X) - \frac{1}{X^2}\right)X = d + \frac{c'X}{(X-1)^2} + \frac{d'X}{(X-1)} \xrightarrow[X\to 0]{} d$$

$$\left(Q(X) - \frac{1}{X^2}\right)X = \frac{3X^2-3X+2}{(X-1)^2} \xrightarrow[X\to 0]{} 2 \qquad \text{donc } d = 2$$

$$R(X) = Q(X) - \frac{1}{X^2} - \frac{2}{X} = \frac{3X^2-3X+2}{X(X-1)^2} - \frac{2(X^2-2X+1)}{X(X-1)^2}$$

$$= \frac{X^2+X}{X(X-1)^2} = \frac{X+1}{(X-1)^2} = \frac{(X-1)+2}{(X-1)^2}$$

$$= \frac{2}{(X-1)^2} + \frac{1}{X-1}$$

$$R(X)(X-1)^2 = c' + d'(X-1) \xrightarrow[X\to 1]{} c'$$

$$R(X)(X-1)^2 = X+1 \xrightarrow[X\to 1]{} 2 \qquad \text{donc } c' = 2$$

$$R(X) - \frac{2}{(X-1)^2} = \frac{X+1-2}{(X-1)^2} = \frac{X-1}{(X-1)^2} = X-1 \qquad \text{donc } d' = 1$$

$$\frac{X^5+1}{X^2(X-1)^2} = X + 2 + \frac{1}{X^2} + \frac{2}{X} + \frac{2}{(X-1)^2} + \frac{1}{X-1}$$

B) $P(X) = \frac{X^4+X+1}{X(X^2+1)^3}$

$$P(X) = \frac{a}{X} + \frac{BX+C}{(X^2+1)^3} + \frac{B'X+C'}{(X^2+1)^2} + \frac{B''X+C''}{(X^2+1)}$$

$P(X)X \xrightarrow[X\to 0]{} a$ et $P(X)X = \frac{X^4+X+1}{(X^2+1)^3} \xrightarrow[X\to 0]{} 1$

donc $a = 1$

$$Q(X) = P(X) - \frac{1}{X} = \frac{X^4 + X + 1 - (X^6 + 3X^4 + 3X^2 + 1)}{X(X^2+1)^3}$$

$$= \frac{-X^6 - 2X^4 - 3X^2 + X}{X(X^2+1)^3} = \frac{-X^5 - 2X^3 - 3X + 1}{(X^2+1)^3}$$

$$\frac{Q(X)X(X^2+1)}{X} \xrightarrow[X \to +\infty]{} B' \quad \text{et} \quad \frac{Q(X)X(X^2+1)}{X} = \frac{-X^5 - 2X^3 - 3X + 1}{X(X^2+1)^2} \xrightarrow[X \to +\infty]{} -1$$

donc $B'' = -1$

$$Q(X) + \frac{X}{(X^2+1)} = \frac{-X^5 - 2X^3 - 3X + 1 + X(X^4 + 2X^2 + 1)}{(X^2+1)^3}$$

$$= \frac{-2X + 1}{(X^2+1)^3}$$

Donc $a = 1$, $B'' = -1$, $C'' = 0$, $B' = C' = 0$, $B = -2$, $C = 1$

$$P(X) = \frac{1}{X} + \frac{-2X+1}{(X^2+1)^3} - \frac{X}{(X^2+1)}$$

$$\frac{1}{X^2+1} = \frac{1}{(X-i)(X+i)} = \frac{a}{X-i} + \frac{B}{X+i} = \frac{(a+B)X + i(-a+B)}{X^2+1}$$

donc $\begin{cases} a+B = 0 \\ i(-a+B) = 1 \end{cases} \Leftrightarrow \begin{cases} B = -a \\ -2ia = 1 \end{cases} \Leftrightarrow 2a = i \Leftrightarrow a = \frac{i}{2}$

$\Leftrightarrow \begin{cases} a = i/2 \\ B = -i/2 \end{cases}$ donc $\frac{1}{X^2+1} = \frac{1}{2}\left(\frac{i}{X+i} - \frac{i}{X-i}\right)$

c) $P(X) = \frac{X^2}{(X-1)(X^2-X+1)^2} =$

$\begin{cases} \deg(\text{Nom}) = 2 < \deg(\text{dénom}) = 5 \\ 1 \text{ racine simple} \\ X^2 - X + 1 \Rightarrow \Delta = 1 - 4 = -3, \text{ pas de racines} \end{cases}$

$$P(X) = \frac{a}{X-1} + \frac{BX+C}{(X^2-X+1)^2} + \frac{B'X+C'}{X^2-X+1}$$

$P(X)(X-1) \xrightarrow[x\to 1]{} a$

$P(X)(X-1) \simeq \frac{X^2}{(X^2-X+1)^2} \xrightarrow[x\to 1]{} 1$ donc $a=1$

$P(X) - \frac{1}{X-1} = \frac{X^2 - (X^4 + X^2 + 1 - 2X^3 + 2X^2 - 2X)}{(X-1)(X^2-X+1)^2}$

$= \frac{-X^4 + 2X^3 - 2X^2 + 2X - 1}{(X-1)(X^2-X+1)^2}$

$$\begin{array}{r|l}
-X^4+2X^3-2X^2+2X-1 & X-1 \\
+X^4-X^3 & \hline -X^3+X^2-X+1 \\
\hline
X^3-2X^2+2X-1 & \\
-X^3+X^2 & \\
\hline
-X^2+2X-1 & \\
+X^2-X & \\
\hline
X-1 & \\
X-1 & \\
\hline
0 &
\end{array}$$

$Q(X) = P(X) - \frac{1}{X-1} = \frac{-X^3+X^2-X+1}{(X^2-X+1)^2}$

$Q(X) \cdot \frac{(X^2-X+1)}{X} \xrightarrow[x\to+\infty]{} B'$

$\frac{Q(X)\cdot(X^2-X+1)}{X} \simeq \frac{-X^3+X^2-X+1}{X(X^2-X+1)} \xrightarrow[x\to+\infty]{} -1$

$Q(X) + \frac{X}{X^2-X+1} \simeq \frac{-X^3+X^2-X+1+X^3-X^2+X}{(X^2-X+1)} = \frac{1}{(X^2-X+1)^2}$

$a=1,\ B'=-1,\ c'=0,\ B=0,\ C=1$

$P(X) = \frac{1}{X-1} + \frac{1}{(X^2-X+1)^2} - \frac{X}{X^2-X+1}$

d) $A(X) = \dfrac{X^4}{X^3-1}$

$X^3-1 = (X-1)(X^2+X+1)$

$P(X) = aX+b + \dfrac{c}{(X-1)} + \dfrac{dx+e}{X^2+X+1}$

car $\deg(\text{num}) - \deg(\text{denom}) = 1$

$P(X) = \dfrac{X^4-X+X}{X^3-1} = \dfrac{X(X^3-1)}{(X^3-1)} + \dfrac{X}{(X^3-1)}$

$= X + \dfrac{X}{X^3-1}$

$Q(X) : P(X) - X = \dfrac{X}{(X^3-1)} = \dfrac{X}{(X-1)(X^2+X+1)}$

$Q(X)(X-1) \xrightarrow[X \to 1]{} c$ et $Q(X)(X-1) = \dfrac{X}{(X^2+X+1)} \xrightarrow[X=1]{} \dfrac{1}{3}$

donc $c = \dfrac{1}{3}$

$Q(X) - \dfrac{1}{3}\dfrac{1}{X-1} = \dfrac{3X-(X^2+X+1)}{3(X-1)(X^2+X+1)} = \dfrac{-X^2+2X-1}{3(X-1)(X^2+X+1)} = \dfrac{-(X-1)^2}{3(X-1)(X^2+X+1)}$

$= \dfrac{-X+1}{3(X^2+X+1)}$

Donc $a=1, b=0, c=\frac{1}{3}, d=-\frac{1}{3}, e=\frac{1}{3}$

Donc $P(X) = X + \frac{1}{3}\dfrac{1}{X-1} + \dfrac{\frac{1}{3}(X+1)}{X^2+X+1}$

Exercice 21 :

Soit $P \in \mathbb{K}[X]$

$P(X) - X \mid P(P(X)) - X$

$\Leftrightarrow P(P(X)) - X \equiv 0 \quad [P(X)-X]$

$\Leftarrow P(X) - X \equiv 0 \quad [P(X)-X]$

Exercice 22

$P \in \mathbb{C}[X]$, $P \notin \mathbb{C}$

P tq $P' \mid P$ sur $\mathbb{C}$, P est scindé

$P = \alpha \prod_i (X - x_i)^{\alpha_i}$ avec $\alpha \in \mathbb{C}^*$, $x_i \in \mathbb{C}$

$P' = a \sum_i \prod_{j \neq i} (X - x_j)^{\alpha_j} \cdot \alpha_i (X - x_i)^{\alpha_i - 1}$, $\alpha_i \in \mathbb{N}^*$

on veut montrer que $P = a(X - x_1)^{\alpha_1}$

si $P = a(X - x)^\alpha$, on a : $P' = a\alpha(X - x)^{\alpha - 1} \mid P$

si x tq $P(x) = 0$

$$P(y) = \sum_{k=0}^{n} \frac{P^{(k)}(x)(y - x)^k}{k!}$$

$$P'(y) = \sum_{k=1}^{n} \frac{P^{(k)}(x)}{k!} \cdot k(y - x)^{k-1}$$