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Exercice 1

$(\mathbb{Z}/13\mathbb{Z})^\times$ de card $\varphi(13) = 12$
$\shortparallel$
$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Montrons que $((\mathbb{Z}/13\mathbb{Z})^\times, \times)$ est cyclique i.e

$\exists a \in (\mathbb{Z}/13\mathbb{Z})^\times, (\mathbb{Z}/13\mathbb{Z})^\times = \langle a \rangle$ ou $o(a) = 12$

$7^1 = 7$ — 7
$7^2 = \overline{49} = \overline{10} = -3$ — 10
$7^3 = \overline{70} = \overline{-21} = \overline{-8} = \overline{5}$ — 5
$7^4 = \overline{5 \cdot 7} = \overline{35} = \overline{-4} = 9$ — 9
$7^5 = -15 = -2 = 11$ — 11
$7^6 = -14 = -1 = 12$ — 12
$7^7 = -7 = 6$ — 6
$7^8 = (-4)(-4) = 16 = 3$ — 3
$7^9 = \overline{21} = 8$ — 8
$7^{10} = (-2)(-2) = 4$ — 4
$7^{11} = (-2)(-1) = 2$ — 2
$7^{12} = 7 \cdot 2 = 14 = 1$ — 1

Donc $o(7) = 12 = |(\mathbb{Z}/13\mathbb{Z})^\times|$

$\Rightarrow (\mathbb{Z}/13\mathbb{Z})^\times = \langle 7 \rangle$
donc cyclique.

b) $|(\mathbb{Z}/12\mathbb{Z})^\times| = \varphi(12) = \varphi(2^2)\varphi(3)$
$= (4-2)(3-1) = 2 \cdot 2 = 4$

$(\mathbb{Z}/12\mathbb{Z})^\times = \{1, 5, 7, 11\}$

$o(11) = 1$ — $\overline{5} \cdot \overline{5} = \overline{25} = \overline{1}$
$\overline{11} = \overline{-1}$ — $o(\overline{5}) = 2$
$\overline{11} \cdot \overline{11} = (-1)(-1) = 1$ — $\overline{7} \cdot \overline{7} = \overline{49} = \overline{1}$
$\Rightarrow o(\overline{7}) = 2$

$o(1) = 1$ — Il n'y a pas d'éléments d'ordre 4, donc le groupe n'est pas cyclique

Exercice 2

$\mathbb{Z}[i] = \{ a+bi : a \in \mathbb{Z}, b \in \mathbb{Z} \}$

a) Soit $a, b \in \mathbb{Z}[i]$

$a + b^{-1} = a + (-b) \in \mathbb{Z}[i]$

donc $(\mathbb{Z}[i], +)$ est un sous-groupe de $(\mathbb{C}, +)$

Soient $a+bi$, $a'+b'i \in \mathbb{Z}[i]$

$(a+bi) \times (a'+b'i) = aa' - bb'i + ab'i + a'bi \in \mathbb{Z}[i]$

$1_{\mathbb{C}} = 1 + 0i \in \mathbb{Z}[i]$ donc $(\mathbb{Z}[i], +, \times)$ est un sous-anneau de $(\mathbb{C}, +, \times)$

$(\mathbb{Z}[i]^{\times}, \times)$ soit $z = 1 + 3i$, cherchons z' tq $zz' = 1$

$z' = \frac{1}{z} = \frac{1-3i}{1+9} = \frac{1}{10} - \frac{3}{10}i \notin \mathbb{Z}[i]$

car $\frac{1}{10} \notin \mathbb{Z}$ et $-\frac{3}{10} \notin \mathbb{Z}$

donc $(\mathbb{Z}[i]^{\times}, \times)$ n'est pas un groupe.
donc n'est pas un sous-corps de $(\mathbb{C}, +, \times)$

b) $z \in \mathbb{Z}[i]$ tq $\frac{1}{z} \in \mathbb{Z}[i]$

$1, -1, i, -i$

Exercice 3

Soit A un anneau intègre i.e $\forall a, b \in A$, $a \cdot b = 0 \Rightarrow a = 0$ ou $b = 0$

Supposons par l'absurde que $A \times A$ est intègre

$0_{A \times A} = (0,0)$ $\quad (a,b) \times (c,d) = (ac, bd)$

Soient $(1,0)$ et $(0,1)$
$\neq (0,0)$ $\quad \neq (0,0)$

$(1,0) \times (0,1) = (1 \cdot 0, 0 \cdot 1) = (0,0)$ mais aucun des facteur ne vaut $(0,0)$. Absurde.

Donc $A \times A$ n'est pas un anneau intègre.

Exercice 4

Soit A un anneau commutatif

$N = \{ a \in A : \exists n \geq 1 \text{ tq } a^n = 0 \}$

a) Soit $a, b \in N$ alors $\exists n, m$ tq $a^n = 0$, $b^m = 0$

$$(a - b)^{nm} = \sum_{k=0}^{nm} \binom{nm}{k} a^k \, b^{nm-k} (-1)^{nm-k}$$

$\forall k \leq m(n-1)$ $\quad b^{nm-k} = 0$

Si $k > m(n-1)$ $\quad a^{m(n-1)} = a^{m} \cdot a^{-m} = (a^n)^m a^{-m} = 0 \cdot a^{-m} = 0$

D'où $(a-b)^{nm} = 0$ donc $a - b \in N$ $\quad \forall a, b \in N$

$\forall a \in N \quad \forall p \in A, \quad \exists n \geq 1$ tq $a^n = 0$

d'où $(ap)^n = a^n p^n$ car A commutatif
$= 0 \cdot p^n = 0$

Donc $a \cdot p \in N$

N vérifie les propriétés d'un idéal donc N est un idéal.

b) Soit $a \in N$ donc $\exists n \geq 1$ $a^n = 0$

$a^n - 1 = -1 = (a-1)(a^{n-1} + a^{n-2} + \dots + 1)$

Donc $1 = (1-a)(a^{n-1} + a^{n-2} + \dots + 1)$

Donc $(1-a)$ est inversible

Si on pose $m = 2n+1$

$a^{2n-1} - 1 = (a-1)(a^{2n} + a^{2n-1} + \dots + 1)$

$a^{2n+1} - 1 = 1 - a^{2n+1}$

$1 + a^{2n+1} = 1 - (-a)^{2n+1} = (1+a)(a^{2n} - a^{2n-1} + \dots + 1)$

Donc $1+a$ est inversible.

c) Soit a inversible et $\exists n \geq 1$ tq $b^n = 0$

$\exists a^{-1}$ tq $a a^{-1} = 1$

$a^{-1}(a+b) = 1 + a^{-1}b$

$a^{-1} \in A$ et $b \in N$
donc $a^{-1}b \in N$ par
def de Ideal

D'après (b) $1 + a^{-1}b$ est
inversible car $a^{-1}b \in N$.

Exercice 8

$$A = X^6 + 1 \qquad \in \mathbb{C}[X]$$

$$B = X^4 + X^3 + X - 1 \qquad \in \mathbb{C}[X]$$

$$
\begin{array}{l|l}
X^6 + 1 & X^4 + X^3 + X - 1 \\ \cline{2-2}
- \; X^6 + X^5 + X^3 - X^2 & X^2 - X + 1 \\ \cline{1-1}
-X^5 - X^3 + X^2 + 1 & \\
-X^5 - X^4 - X^2 + X & \\ \cline{1-1}
X^4 - X^3 + 2X^2 - X + 1 & \\
X^4 + X^3 + X - 1 & \\ \cline{1-1}
-2X^3 + 2X^2 - 2X + 2 &
\end{array}
$$

b)
$$
\begin{array}{l|l}
X^4 + X^3 + X - 1 & -2X^3 + 2X^2 - 2X + 2 \\ \cline{2-2}
X^4 - X^3 + X^2 - X & -\frac{1}{2}X - 1 \\ \cline{1-1}
2X^3 - X^2 + 2X - 1 & \\
2X^3 - 2X^2 + 2X - 2 & \\ \cline{1-1}
X^2 + 1 &
\end{array}
$$

$$
\begin{array}{l|l}
-2X^3 + 2X^2 - 2X + 2 & X^2 + 1 \\ \cline{2-2}
-2X^3 \qquad\quad - 2X & -2X + 2 \\ \cline{1-1}
\qquad - \; 2X^2 \qquad + 2 & \\
\qquad\quad\; 2X^2 \qquad + 2 & \\ \cline{1-1}
\qquad\qquad\qquad\quad 0 &
\end{array}
$$

Donc $A = (X^4 + X^3 + X - 1)C + \underbrace{X^2 + 1}_{D}$

$$D = X^2 + 1$$

c) $X^6+1 = (X^4+X^3+X-1)(X^2-X+1) + (-2X^3+2X^2-2X+2)$

$(X^4+X^3+X-1) = (-2X^3+2X^2-2X+2)(-\frac{1}{2}X-1) + X^2+1$

$(-2X^3+2X^2-2X+2) = (-2X+2)(X^2+1) + 0$

$X^2+1 = (X^4+X^3+X-1) + (-2X^3+2X^2-2X+2)(-\frac{1}{2}X-1)(-1)$

$X^2+1 = (X^4+X^3+X-1) + \left((X^6+1)+(X^4+X^3+X-1)(-X^2+X-1)\right)(\frac{1}{2}X+1)$

$X^2+1 = (X^6+1)\underbrace{(\frac{1}{2}X+1)}_{u_0} + (X^4+X^3+X-1)\underbrace{(-\frac{1}{2}X^3-\frac{1}{2}X^2+\frac{1}{2}X)}_{V_0}$

$(-X^2+X-1)(\frac{1}{2}X+1)$

$= -\frac{1}{2}X^3+\frac{1}{2}X^2-\frac{1}{2}X-X^2+X-1$

$= -\frac{1}{2}X^3-\frac{1}{2}X^2+\frac{1}{2}X-1$

$D = AU + BV$

$0 = A(U-U_0) + B(V-V_0)$

$\begin{cases} \tilde{A} = \frac{A}{D} \\ \tilde{B} = \frac{B}{D} \end{cases}$ on fait ça pour avoir $\tilde{A} \wedge \tilde{B} = 1$

$\Leftrightarrow 0 = \tilde{A}(U-U_0) + \tilde{B}(V-V_0)$

$\Rightarrow -\tilde{A}\underbrace{(U-U_0)}_{\tilde{u}} = \tilde{B}\underbrace{(V-V_0)}_{\tilde{v}}$

$\tilde{u} = \tilde{B}P$
$\tilde{v} = -\tilde{A}P$ $\Rightarrow$ $\begin{cases} U = \tilde{B}P + U_0 \\ V = -\tilde{A}P + V_0 \end{cases}$

Exercice 10

$X^2 \equiv 1 \mod X^2-1$

$X^{2n} = (X^2)^n \equiv 1 \mod X^2-1$

$X^{21} = X^{20}X \equiv X \mod X^2-1$

$X^{19} = X^{18}X \equiv X \mod X^2-1$
$X^{17} = X^{16}X \equiv X \mod X^2-1$

Donc $X^{21}+X^{19}+X^{17} \equiv 3X \mod X^2-1$

B) $X \equiv 1 \mod X-1$

$X^{45} \equiv X \mod X-1$

$X^{43} \equiv X \mod X-1$

$X^{45}-X^{43}+1 \equiv 1 \mod X-1$

$$X^{45}-X^{43}+1-1 = X^{45}-X^{43} = \frac{X^{43}(X^2-1)}{X-1}$$

$$= \frac{X^{43}(X-1)(X+1)}{X-1}$$

$$= X^{43}(X+1) = X^{44}+X^{43}$$

$\overbrace{(X^{44}+X^{43})}^{C}(X-1) + \underbrace{1}_{R}$

$X^{45}+X^{44}-X^{44}-X^{43}+1 = X^{45}-X^{43}+1$
$(X^{44}+X^{43}) \equiv 2 \mod X-1$

$$X^{45}-X^{43}+1 = (X^{44}+X^{43})(X-1)+1 \mod (X-1)^2$$
$$\equiv ((X-1)Q+2)(X-1)+1$$
$$\equiv (X-1)^2Q+2X-2+1$$
$$\equiv 2X-1 \mod (X-1)^2$$

c) $X^{36} - X^{23}$ par X^2+1

$X^2 \equiv -1 \mod X^2+1$

$X^{36} = (X^2)^{18} \equiv (-1)^{18} \equiv 1 \mod X^2+1$

$X^{23} = (X^{22})X = (X^2)^{11}X \equiv -X \mod X^2+1$

$1-(-X) = \underline{\underline{1+X}}$

<u>Exercice 12</u>

a) $X^3 - X^2 + X - 1$
$= X(X^2 - X + 1) - 1$
$= X((X-1)^2 + X) - 1$
$= X(X-1)^2 + X^2 - 1$
$= X(X-1)^2 + (X-1)(X+1)$
$= (X-1)(X(X-1) + X + 1)$
$= (X-1)(X^2 - X + X + 1)$
$= (X-1)(X^2+1) = P$

b) $Q_n = (X^2 - X + 1)^n - X^{2n} + X^n - 1$

$P \mid Q_n \Rightarrow (X-1) \mid Q_n$ et $(X^2+1) \mid Q_n$ car $(X-1) \wedge X^2+1 = 1$

$Q_n(1) = (1-1+1)^n - 1 + 1 - 1 = 1 - 1 + 1 - 1 = 0$

$Q_n \equiv (X^2 - X + 1)^n - X^{2n} + X^n - 1 \mod X^2+1$
$\equiv (-X)^n - (-1)^n + (X)^n - 1 \mod X^2+1$

Si $n = 2n'+1$ alors : $Q_n \equiv -X - (-1) + X - 1$
$\equiv -X + X + 1 - 1 = 0$

Si $n = 2n'$
$(-X)^{2n'} - X^{2\cdot 2n'} + X^{2n'} - 1$

$X^n((-1)^n + 1) - (-1)^n - 1$
$X^{2n'}(2) - 1 - 1$
$= (-1)^{n'}(2) - 2$ si $n' = 2n''$
$= 0$

Donc $P \mid Q_n$ pour n impair et $4 \mid n$

Exercice 21

Montrons que $P(X) - X \mid P(P(X)) - X$

$P(X) \equiv X \mod P(X) - X$

Donc $P(P(X)) - X \equiv P(X) - X \mod P(X) - X$
$\equiv 0 \mod P(X) - X$

Donc $P(X) - X \mid P(P(X)) - X$

Exercice 22

$P \in \mathbb{C}[X]$ tq $P' \mid P$

Si $P' \mid P$ alors P' et P partagent les mêmes racines

$P = d(aX+b)^k$

$P' = ak\, d(aX+b)^{k-1}$

$P = Q P' = (qX+h)\, ak d(aX+b)^{k-1}$

$\Rightarrow \quad P' = \dfrac{P}{Q} = \dfrac{d(aX+b)^k}{qX+h} = akd(aX+b)^{k-1}$

$\Rightarrow \quad \dfrac{(aX+b)}{qX+h} = ak$

$\Rightarrow \quad qX + h = \frac{1}{k}X + \frac{b}{k}$

$\Rightarrow \quad \dfrac{aX+b}{\frac{1}{k}X + \frac{b}{k}} = ak$

Exercice 15

Soient $m, n \in \mathbb{N}^*$ et $d := m \wedge n$

Soit $z \in \mathbb{C}$

Supposons que $z^n = z^m = 1$

$\exists\, n', m'$ tq $n = n'd$
$m = m'd$

donc $z^{n'd} = z^{m'd} = 1$

$\Rightarrow (z^d)^{n'} = (z^d)^{m'} = 1$

$\Rightarrow z^d = 1^{\frac{1}{n'}} = 1 = 1^{\frac{1}{m'}}$

$\Rightarrow \underline{\underline{z^d = 1}}$

$X^m - 1 = \prod_{z \in \mathcal{U}_m} (X - z) \qquad |\mathcal{U}_m| = m$

$X^n - 1 = \prod_{z \in \mathcal{U}_n} (X - z) \qquad |\mathcal{U}_n| = n$

$X^d - 1 = \prod_{z \in \mathcal{U}_d} (X - z) \qquad |\mathcal{U}_d| = d$

$(X^m - 1) \wedge (X^n - 1) = \prod_{z \in \mathcal{U}_m \cap \mathcal{U}_n} (X - z) = \prod_{z \in \mathcal{U}_d} (X - z) = X^d - 1$

6)

On suppose que $m \geq n$ et $m = nq + r$

$X^m - 1 \equiv \qquad \text{mod } X^n - 1$

$X^{nq+r} - 1 \equiv X^{nq} X^r - 1 \equiv X^r - 1 \text{ mod } X^n - 1$ car $X^n \equiv 1 \text{ mod } X^n - 1$

$X^m - 1 \wedge X^n - 1 = X^n - 1 \wedge X^r - 1 = \ldots = X^d - 1$

C) $P \in \mathbb{C}[X]$

(i) D'après (a) $P^m - 1 \wedge P^n - 1 = P^d - 1$

(ii) Donc $\exists U, V \in \mathbb{C}[X]$ tq $U(P^m - 1) + V(P^n - 1) = P^d - 1$

Supposons que $d = m \wedge n = 1 \Rightarrow U'(P^m - 1) + V'(P^n - 1) = P - 1$

$P^m - 1 = (P-1)Q$ $\qquad$ $P^n - 1 = (P-1)Q'$

$(P^d)^{m'} - 1 = (P^d - 1)Q$ $\qquad$ $(P^d)^{n'} - 1 = (P^d - 1)Q'$

$P^{mn} - 1 = (P-1)Q''$ $\qquad$ $= (P^m)^n - 1 = (P^m - 1)H$

$\qquad = (P^n)^m - 1 = (P^n - 1)H'$

$\Rightarrow P - 1 \mid P^{mn} - 1$

$\Rightarrow P^n - 1 \mid P^{mn} - 1$

$\Rightarrow P^m - 1 \mid P^{mn} - 1$

$$\left.\begin{array}{l} P-1 \mid P^{mn}-1 \\ P-1 \mid P^m - 1 \\ P^m - 1 \mid P^{mn} - 1 \end{array}\right\} \Rightarrow \frac{P^m - 1}{P-1} \,\Big|\, \frac{P^{mn} - 1}{P-1}$$

$$\left.\begin{array}{l} P-1 \mid P^{mn}-1 \\ P-1 \mid P^n - 1 \\ P^n - 1 \mid P^{mn} - 1 \end{array}\right\} \Rightarrow \frac{P^n - 1}{P-1} \,\Big|\, \frac{P^{mn} - 1}{P-1}$$

$\dfrac{P^m - 1}{P-1} \wedge \dfrac{P^n - 1}{P-1} = 1$ car $P - 1$ est leur pgcd

donc $\left(\dfrac{P^m - 1}{P-1} \dfrac{P^n - 1}{P-1} \,\Big|\, \dfrac{P^{mn} - 1}{P-1} \right) \cdot (P-1)^2 \Rightarrow (P^m - 1)(P^n - 1) \mid (P^{mn} - 1)(P - 1)$

Exercice 27

a) $\frac{X^5+1}{X^2(X-1)^2}$ $\qquad \underset{=5}{\deg(X^5+1)} > \underset{=4}{\deg(X^2(X-1)^2)}$

Donc il faut effectuer euclidivision.

$X^5+1=0 \Rightarrow X^5=-1 \Rightarrow X=-1$

Donc $X^5+1=(X-1)Q$

$$\begin{array}{l|l} X^5+1 & X^4-2X^3+X^2 \\ \cline{2-2} \underline{X^5-2X^4+X^3} & X+2 \\ +2X^4-X^3+1 & \\ \underline{2X^4-4X^3+2X^2} & \\ 3X^3-2X^2+1 & \end{array}$$

$$\frac{X^5+1}{X^4-2X^3+2X} = \frac{3X^3-2X+1}{X^2(X-1)^2} + X+2$$

$$\frac{3X^3-2X^2+1}{X^2(X-1)^2} = \frac{a}{X} + \frac{b}{X^2} + \frac{c}{X-1} + \frac{d}{(X-1)^2}$$

$$\frac{3X^3-2X^2+1}{X^2} = \frac{a(X-1)^2}{X} + \frac{b(X-1)^2}{X^2} + c(X-1) + d$$

$$\frac{3-2+1}{1} = d \Rightarrow d=2$$

$$\frac{3X^3-2X^2+1}{(X-1)^2} = aX + b + \frac{cX^2}{X-1} + \frac{dX^2}{(X-1)}$$

$$\frac{1}{1} = b \Rightarrow b=1$$

$$P(X) = \frac{3X^3-2X^2+1}{X^2(X-1)^2} = \frac{a}{X} + \frac{1}{X^2} + \frac{c}{X-1} + \frac{2}{(X-1)^2}$$

$$P(X)X = \frac{3X^2-2X+\frac{1}{X}}{(X-1)^2} = a + \frac{1}{X} + \underbrace{\frac{cX}{(X-1)}}_{\frac{c}{1-\frac{1}{X}}} + \frac{2X}{(X-1)^2} \xrightarrow[X\to+\infty]{} a+c$$

$$\frac{3X-2+\frac{1}{X^2}}{X-2+\frac{1}{X}} = \frac{3-\frac{2}{X}+\frac{1}{X^3}}{1-\frac{2}{X}+\frac{1}{X^2}} \xrightarrow[X\to+\infty]{} 3$$

donc $\quad a + c = 3$

$$P(x) = \frac{3x^3 - 2x^2 + 1}{x^2(x-1)^2} = \frac{a}{x} + \frac{1}{x^2} + \frac{c}{x-1} + \frac{2}{(x-1)^2}$$

$$\frac{3x^3 - 2x^2 + 1}{x^2(x-1)^2} - \frac{2}{(x-1)^2} = \frac{3x^3 - 2x^2 + 1 - 2x^2}{x^2(x-1)^2}$$

$$= \frac{3x^3 - 4x^2 + 1}{x^2(x-1)^2}$$

$$\frac{3x^3 - 4x^2 + 1}{x^2(x-1)} = \frac{a}{x}(x-1) + \frac{1}{x^2}(x-1) + c$$

$$3x^3 - 4x^2 + 1 = ax(x-1)^2 + (x-1)^2 + cx^2(x-1)$$

$$3x^3 - 4x^2 + 1 = ax^3 - 2ax^2 + ax + x^2 - 2x + 1 + cx^3 - cx^2$$

$$= x^3(a+c) + x^2(-2a+1-c) + x(a-2) + 1$$

$$\Rightarrow \begin{cases} a + c = 3 \\ -2a + 1 - c = 4 \\ a - 2 = 0 \end{cases} \quad \Rightarrow c = 1, \quad \Rightarrow a = 2$$

Donc

$$P(x) = \frac{3x^3 - 2x^2 + 1}{x^2(x-1)^2} = \frac{2}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

$$\frac{x^5 + 1}{x^4 - 2x^3 + 2x} = \frac{3x^3 - 2x + 1}{x^2(x-1)^2} + x + 2$$

$$= x + 2 + \frac{2}{x} + \frac{1}{x^2} + \frac{1}{x-1} + \frac{2}{(x-1)^2}$$

b)

$$P(X) = \frac{X^4+X+1}{X(X^2+1)^3}$$

Nominateur est de degré < degré de dénominateur

D'après le théorème de décomposition en éléments simples
$\exists\ \lambda, a, b, c, d, e, f \in \mathbb{R}$ tq

$$P(X) = \frac{\lambda}{X} + \frac{aX+b}{X^2+1} + \frac{cX+d}{(X^2+1)^2} + \frac{eX+f}{(X^2+1)^3}$$

$$P(X)X = \lambda + X\left(\frac{aX+b}{X^2+1} + \frac{cX+d}{(X^2+1)^2} + \frac{eX+f}{(X^2+1)^3}\right)$$

$$= \frac{X^4+X+1}{(X^2+1)}$$

On pose $X=0 \Rightarrow P(X)X = \frac{1}{1} = 1 = \lambda \Rightarrow \underline{\lambda = 1}$

$$P(X)(X^2+1)^3 - \frac{1}{X}(X^2+1)^3 = (aX+b)(X^2+1)^2 + (cX+d)(X^2+1) + eX+f$$

$$= \frac{X^4+X+1}{X} - \frac{(X^2+1)^3}{X}$$

$$= \frac{-X^6-2X^4-3X^2+X}{X}$$

$$= -X^5-2X^3-3X+1 = aX^5+2aX^3+aX+\underline{bX^4}+2bX^2+b$$
$$+cX^3+cX+dX^2+d$$
$$+eX+f$$

$$= X^5(a) + X^4(b) + X^3(2a+c) + X^2(2b+d)$$
$$+X(a+c+e)$$
$$+b+d+f$$

$$\Rightarrow \begin{cases} \underline{a=-1} \\ \underline{b=0} \\ 2a+c=-2 \Rightarrow c=-2-2a=-2-2(-1)=-2+2=0 \Rightarrow \underline{c=0} \\ 2b+d=0 \Rightarrow d=0 \\ a+c+e=-3 \Rightarrow e=-3-a-c = -3+1=-2 \Rightarrow \underline{e=-2} \\ b+d+f=1 \Rightarrow f=1 \end{cases}$$

<u>Donc:</u>

$$P(X) = \frac{1}{X} - \frac{X}{X^2+1} + \frac{-2X+1}{(X^2+1)^3} = \frac{X^4+X+1}{X(X^2+1)^3}$$

$$\frac{(X^2+1)^3 - X^2(X^2+1)^2 - 2X^2 + X}{X(X^2+1)^3}$$

$$(X^4+2X^2+1)(X^2+1) =$$

$$\cancel{X^6} + \cancel{2X^4} + \cancel{X^2} + \underline{X^4} + \cancel{2X^2} + 1 - \cancel{X^6} - \cancel{2X^4} - \cancel{X^2} - \cancel{2X^2} + X$$

$$= X^4 + X + 1$$

<u>OK!</u>

c)

$$\frac{X^2}{(X-1)(X^2-X+1)^2} =: P(X)$$

$$\deg(X^2) < \deg\left((X-1)(X^2-X+1)^2\right)$$

donc d'après le théorème de décomposition en éléments simples

$\exists \lambda, a, b, c, d \in \mathbb{R}$

$$P(X) = \frac{\lambda}{X-1} + \frac{aX+b}{X^2-X+1} + \frac{cX+d}{(X^2-X+1)^2}$$

$$= \frac{\lambda(X^2-X+1)^2 + (aX+b)(X^2-X+1)(X-1) + cX+d(X-1)}{(X-1)(X^2-X+1)^2}$$

$$\Rightarrow X^2 = \lambda X^4 - \lambda X^3 + \lambda X^2 - \lambda X^3 + \lambda X^2 - \lambda X + \lambda X^2 - \lambda X + \lambda$$
$$+ aX^4 - 2aX^3 + 2aX^2 - aX + bX^3 - 2bX^2 + 2bX - b$$
$$+ cX + dX - d$$
$$= X^4(\lambda + a) + X^3(-2\lambda - 2a + b) + X^2(3\lambda + 2a - 2b) + X(-2\lambda - a + 2b + c + d)$$
$$(\lambda - b - d)$$

$\Rightarrow$

$$\begin{cases} \lambda + a = 0 & \Rightarrow a = -1 \\ -2\lambda - 2a + b = 0 & \Rightarrow b = 0 \\ 3\lambda + 2a = 1 & \Rightarrow \lambda = 1 \\ -2 + 1 + c + d = 0 & \Rightarrow c = -1 + 1 = 0 \\ 1 - d = 0 & \Rightarrow d = 1 \end{cases}$$

Donc

$$P(X) = \frac{1}{X-1} - \frac{X}{X^2-X+1} + \frac{1}{(X^2-X+1)^2}$$

$$(X^2-X+1)^2 - (X^2-X+1)(X-1)X + (X-1)$$

$$(X^2-X+1)^2(1-X^2+X) \quad +(X+1)$$

d) $\dfrac{X^4}{X^3-1}$

$\deg(X^4) > \deg(X^3-1)$ Alors, effectuons la division euclidienne pour avoir une partie entière

$$\begin{array}{l|l} X^4 & X^3-1 \\ \cline{2-2} X^4-X & X \\ \cline{1-1} X & \end{array}$$

$$X^4 = (X^3-1)X + X$$

$$\Rightarrow \frac{X^4}{X^3-1} = X + \frac{X}{X^3-1}$$

X^3-1 n'est pas en forme reductible

$X^3-1 = 0 \Rightarrow X^3 = 1 \Rightarrow X = 1$

Donc $(X-1) \mid X^3-1$

$$\begin{array}{l|l} X^3-1 & X-1 \\ \cline{2-2} X^3-X^2 & X^2+X+1 \\ \cline{1-1} X^2-1 & \\ X^2-X & \\ \cline{1-1} X-1 & \\ X-1 & \\ \cline{1-1} 0 & \end{array}$$

Donc $(X^3-1) = (X-1)(X^2+X+1)$

$X^2+X+1 = 0$

$\Delta = 1-4 = -3$

Donc il est en forme réductible

$$P(X) := \frac{X}{(X-1)(X^2+X+1)}$$

$$\frac{X^4}{X^3-1} = X + \frac{X}{X^3-1}$$

$$= X + P(X)$$

$$= X + \frac{X}{(X-1)(X^2+X+1)}$$

$$P(X) = \frac{X}{(X-1)(X^2+X+1)}$$

Dénominateur est en forme irreductible

de plus deg(nom) < deg(dénom)
donc d'après le théorème de décomposition
en éléments simples

$\exists\, \lambda, a, b \in \mathbb{R}$

$$P(X) = \frac{\lambda}{X-1} + \frac{aX+b}{X^2+X+1}$$

$$P(X)(X-1) = \frac{X}{X^2+X+1} = \lambda + \left(\frac{aX+b}{X^2+X+1}\right)(X-1)$$

Posons $X=1$ et on obtient $\frac{1}{3} = \lambda \Rightarrow \lambda = \frac{1}{3}$

$$P(X) = \frac{\frac{1}{3}X^2 + \frac{1}{3}X + \frac{1}{3} + aX^2 + bX - aX - b}{(X-1)(X^2+X+1)}$$

$$= \frac{X^2\left(\frac{1}{3}+a\right) + X\left(\frac{1}{3}+b-a\right) + \left(\frac{1}{3}-b\right)}{(X-1)(X^2+X+1)}$$

$$\Rightarrow \begin{cases} \frac{1}{3}+a=0 \Rightarrow a = -\frac{1}{3} \\ \frac{1}{3}+b-a=1 \Rightarrow b = 1-\frac{1}{3}-\frac{1}{3} = 1-\frac{2}{3} = \frac{1}{3} \\ \frac{1}{3}-b=0 \Rightarrow b=\frac{1}{3} \end{cases}$$

Donc $P(X) = \dfrac{\frac{1}{3}}{X-1} + \dfrac{-\frac{1}{3}X+\frac{1}{3}}{X^2+X+1}$

Donc

$$\frac{x^4}{x^3-1} = x + \frac{x}{x^3-1} = x + \frac{\frac{1}{3}}{x-1} + \frac{-\frac{1}{3}x+\frac{1}{3}}{x^2+x+1}$$