r.v $\quad X : \Omega \to \mathbb{R}$ → distribution

$P(X \dots) = ?$

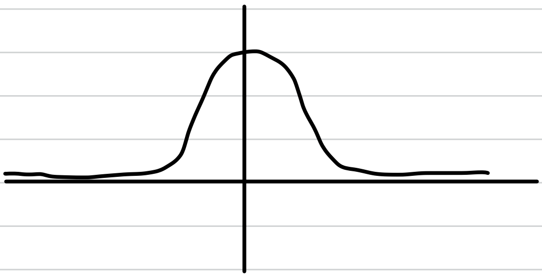
How likely to happen?

PMF — point mass, one — $P(X=x)$

around PDF — point, density — $P(a \le X \le b)$

CDF — $P(X \le x)$

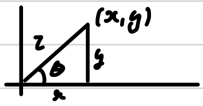
Normal distribution



$N(0,1)$ has PDF $\quad f(z) = c\, e^{-\frac{z^2}{2}}$

mean, variance

$\sqrt{2\pi}$ ↳ normalizing constant such that area to be $\underline{\underline{1}}$



$r^2 = x^2 + y^2$

How to calculate c?

$$\int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\, dz \int_{-\infty}^{+\infty} e^{-\frac{z^2}{2}}\, dz = \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx \int_{-\infty}^{+\infty} e^{-\frac{y^2}{2}}\, dy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(x^2+y^2)/2}\, dx\, dy = \underbrace{\int_0^{2\pi}\int_0^{+\infty} e^{-r^2/2}\, r\, dr\, d\theta}_{u = r^2/2 \quad du = r\,dr} \quad \text{↳ Jacobian}$$

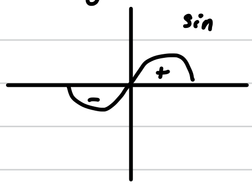
$$= \int_0^{2\pi}\left(\underbrace{\int_0^{\infty} e^{-u}\, du}_{1} \right) d\theta = 2\pi \Rightarrow \int_{-\infty}^{+\infty} e^{-z^2/2}\, dz = \sqrt{2\pi}$$

then $c = \sqrt{2\pi}$

Mean

$Z \sim N(0,1)$, $E[Z] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} z\, e^{-z^2/2} dz = 0$ (by symmetry)

(if $g(x)$ is odd, i.e. $g(-x) = -g(x)$ then $\int_{-a}^{a} g(x) dx = 0$)



Variance

$$Var(Z) = E[Z^2] - \underbrace{E[Z]^2}_{=0^2=0} = E[Z]^2$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \underbrace{z^2 e^{-\frac{z^2}{2}}}_{\text{even function}} dz = \frac{2}{\sqrt{2\pi}} \int_0 \underbrace{z}_{u} \underbrace{z e^{-z^2/2} dz}_{dv}$$

$u = z$ $\quad dv = z e^{-\frac{z^2}{2}}$

$du = dz$ $\quad v = -e^{-\frac{z^2}{2}}$

$$= \frac{2}{\sqrt{2\pi}} \Big(\underbrace{[uv]_0^{\infty}}_{0} + \int_0^{+\infty} e^{-z^2/2} dz = 1$$

<u>Notation:</u> Φ is the standard normal CDF

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2} dt$$

Let $X = \mu + \sigma Z^{\sim N(0,1)}$, $\mu \in \mathbb{R}$ (mean, location)
$\sigma > 0$ (SD, scale)
écart type

Then we say $X \sim N(\mu, \sigma^2)$

$E[X] = \mu$: $E[\mu + \sigma Z] = \underset{\mu}{E[\mu]} + \underset{\sigma E[Z] = 0}{E[\sigma Z]}$

$Var(X) = Var(\mu + \sigma Z) = \sigma^2 \underbrace{Var(Z)}_{1} = \sigma^2$

$Var(X + c) = Var(X)$
$Var(cX) = c^2 \, Var(X)$

$Var(X) > 0$
$Var(X) = 0$ iff $P(X = a) = 1$ for some constant a.

$Var(X + Y) \neq Var(X) + Var(Y)$ proof: $Var(X + X) = Var(2X) = 4Var(X)$
But if X, Y independent, $Var(X + Y) = Var(X) + Var(Y)$

$Z = \frac{X - \mu}{\sigma}$ (standardization)

PDF of $N(\mu, \sigma^2) \sim X$

CDF: $P(X \le x) = P\left(\frac{X - \mu}{\sigma} \le \frac{x - \mu}{\sigma}\right) = \Phi\left(\frac{x - \mu}{\sigma}\right)$

$\Rightarrow$ PDF $\frac{1}{\sigma\sqrt{2\pi}} e^{-\left(\frac{x - \mu}{\sigma}\right)^2 / 2}$

$-X = -\mu + \sigma(-Z) \sim N(-\mu, \sigma^2)$

$68 - 95 - 99.7\%$ Rule

$$P(|X-\underset{\text{mean}}{\mu}| \le \underset{\text{SD}}{\sigma}) \approx 0.68$$

proba that value is not far from mean, less then SD

([illegible])

$$P(|X-\mu| \le 2\sigma) \approx 0.95$$

$$P(|X-\mu| \le 3\sigma) \approx 0.997$$

$$\begin{array}{l} \quad\;\; P_0 \;\; P_1 \;\; P_2 \;\; P_3 \; \ldots \\ X : 0, 1, 2, 3, \ldots \end{array}$$

$$x^2 : 0^2, 1^2, 2^2, 3^2 \ldots\ldots$$

$$E[X] = \sum_x x P(X=x)$$

$$E[X] = \sum_x x^2 P(X=x)$$

$X \sim Pois(\lambda)$

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{\lambda}$$

$$\lambda \sum_{k=1}^{\infty} \frac{k\lambda^{k-1}}{k!} = \lambda e^{\lambda}$$

$$\sum_{k=1}^{\infty} \frac{k\lambda^k}{k!} = \lambda e^{\lambda}$$

$$\sum_{k=1}^{\infty} \frac{k^2 \lambda^{k-1}}{k!} = \lambda e^{\lambda} + e^{\lambda} = e^{\lambda}(\lambda+1)$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} e^{\lambda} \lambda(\lambda+1) = \lambda^2 + \lambda$$

$$Var(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

Prove LOTUS for discrete sample space

Show $E[g(X)] = \sum_x g(x) P(X=x)$

$$\underbrace{\sum_x g(x) P(X=x)}_{\text{grouped}} = \underbrace{\sum_{s \in S} g(X(s)) P(\{s\})}_{\text{ungrouped}}$$

$$\sum_x \sum_{s: X(s)=x} \overbrace{g(X(s))}^{g(x)} P(\{s\}) = \sum_x g(x) \underbrace{\sum_{s: X(s)=x} P(\{s\})}_{P(X=x)}$$

$$= \sum_x g(x) P(X=x)$$