Inequalities

1. Cauchy - Schwartz

$|E(XY)| \leq \sqrt{E(X^2)E(Y^2)}$

[If X, Y uncorrelated (independent) i.e $E[XY] = E[X]E[Y]$]

Ex: $E(X \cdot 1) \leq \sqrt{E(X^2)E(1)} = \sqrt{E(X^2)}$

$\Leftrightarrow \quad E(X)^2 \leq E(X^2)$

Remind: $Var(X) = E(X^2) - E(X)^2$ } Variance can't be negative

Interpretation:

If X and Y have mean 0, then

$$\left|Corr(X, Y)\right| = \left|\frac{E(XY)}{\left(E(X^2)\,E(Y^2)\right)^{1/2}}\right| \leq 1$$

2. Markov

$P(|X| \geq a) \leq \frac{E[|X|]}{a}$, for any $a > 0$

$\Updownarrow$

$E[\mathbb{1}_{|X|\geq a}]$

$a\,\mathbb{1}_{|X|\geq a} \leq |X|$, So $a\,E[\mathbb{1}_{|X|\geq a}] \leq E[|X|]$

$\Leftrightarrow$

$$\mathbb{1}_{|X|\geq a} = \begin{cases} 1 & \text{if } |X| \geq a \Rightarrow \frac{|X|}{a} \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

intuition

Ex: Suppose we have 100 people. Is it possible that at least 95% younger than avg in the group.

...... 50% are older than twice average.

3. Chebyshev

Let $\mu = E[X]$ and $\sigma = \sqrt{Var[X]}$: SD

$$P(|X-\mu| > a) \le \frac{Var(X)}{a^2}, \quad a > 0$$

$$P(|X-\mu| > c\sigma) \le \frac{1}{c^2}$$ Because $Var(X) = \sigma^2$ then $\frac{\sigma^2}{c^2\sigma^2} = \frac{1}{c^2}$

proof: $P(|X-\mu|>a) = P((X-\mu)^2 > a^2) \le \frac{E[(X-\mu)^2]}{a^2} \overset{\text{def}}{=} \frac{Var(X)}{a^2}$

Law of Large Numbers

Let $X_1, X_2, \dots$ Be i.i.d (independent, identical, distributed), mean μ, Var σ^2,

Let $\overline{X_n} = \frac{1}{n}\sum_{j=1}^{n} X_j$ (sample mean) (take first n data and calculate mean)

Law of large numbers: $\overline{X}_n \xrightarrow[n\to\infty]{} \mu$ with probability 1

sample mean converges to the true mean