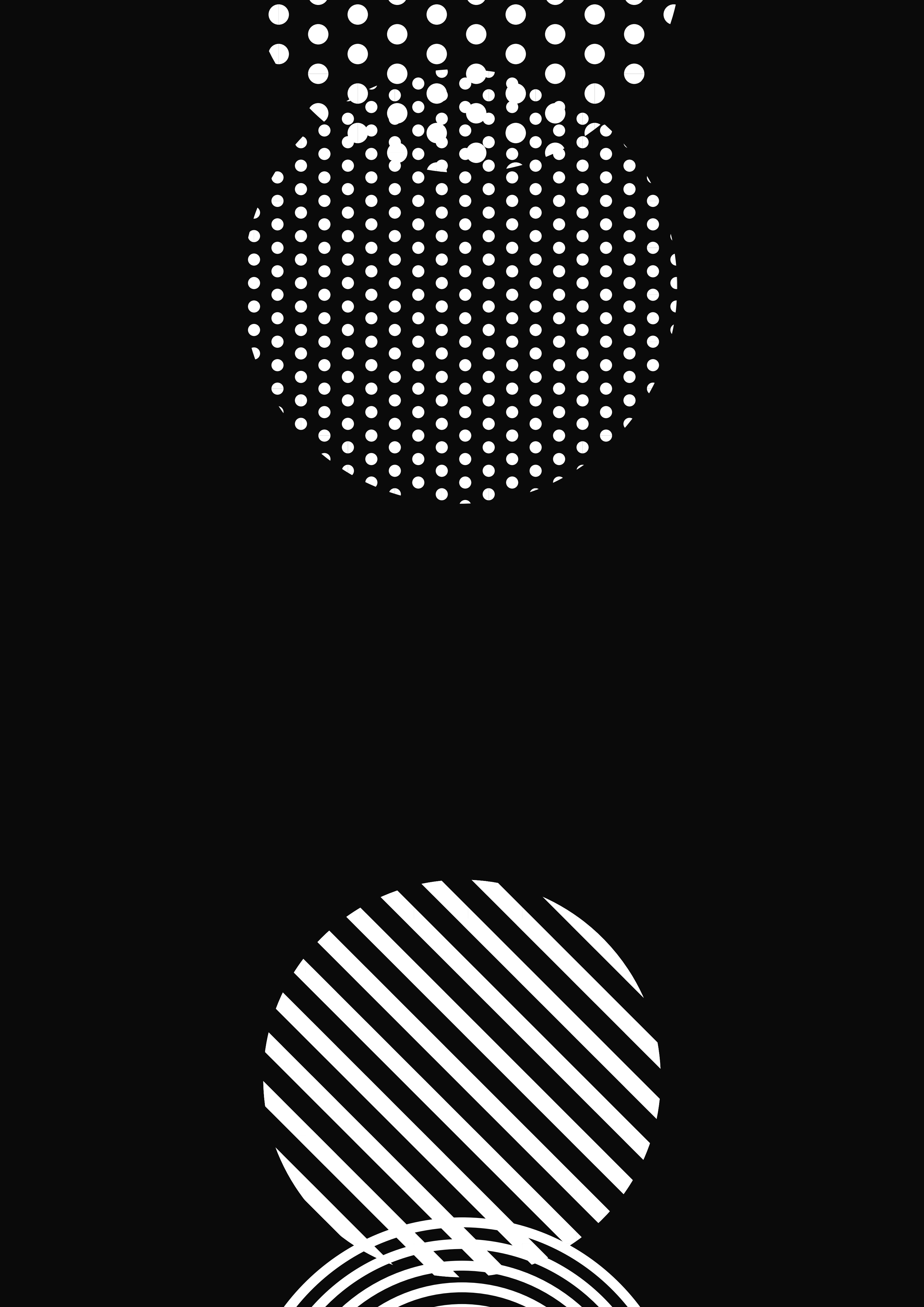

Exercice 1

a) $\sum \frac{1}{\sqrt{n}} x^n$

$$\left| \frac{x^n x}{\sqrt{n+1}} \frac{\sqrt{n}}{x^n} \right| = x \frac{\sqrt{n}}{\sqrt{n+1}} < 1$$

$$x < \sqrt{\frac{n+1}{n}}$$

$$x < \sqrt{1 + \frac{1}{n}} \xrightarrow[n \to +\infty]{} 1$$

Donc $R = 1$

b) $\sum \frac{(-1)^n}{3^n+1} x^{2n+1}$

$$\left| \frac{\cancel{x^{2n+1}} x^2}{3^n 3+1} \frac{3^n+1}{\cancel{x^{2n+1}}} \right| = x^2 \frac{3^n+1}{3^n 3+1} < 1$$

$$\Rightarrow x^2 < \frac{3^n 3+1}{3^n+1}$$

$$\Rightarrow x^2 < \frac{\cancel{3^n}}{\cancel{3^n}} \frac{3+\frac{1}{3^n}}{1+\frac{1}{3^n}} \xrightarrow[n \to +\infty]{} 3$$

$$= 3$$

$$\Rightarrow x < \sqrt{3}$$

Donc $R = \sqrt{3}$

c) $\sum (2^n - n) x^n$

$$\left| \frac{(2^n 2 - n - 1) \cancel{x^n} x}{(2^n - n) \cancel{x^n}} \right| = x \frac{2^n 2 - n - 1}{2^n - n} < 1$$

$$\Rightarrow x < \frac{2^n - n}{2^n 2 - n - 1}$$

$$\Rightarrow x < \frac{1 - \frac{n}{2^n}}{2 - \frac{n}{2^n} - \frac{1}{2^n}} \xrightarrow[n \to +\infty]{} \frac{1}{2}$$

$$x < \frac{1}{2}$$

$R = \frac{1}{2}$

d) $\sum \left(\tan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right) x^n$

$$\left| \frac{\left(\tan\left(\frac{1}{n+1}\right) - \sin\left(\frac{1}{n+1}\right)\right) x^n x}{\left(\tan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)\right) x^n} \right| \Rightarrow x < \frac{\tan\left(\frac{1}{n}\right) - \sin\left(\frac{1}{n}\right)}{\tan\left(\frac{1}{n+1}\right) - \sin\left(\frac{1}{n+1}\right)} \underset{n \to +\infty}{\sim} 1$$

Donc $R = 1$

e) $\sum e^{-n^2} x^n$

$$\left| e^{-n^2} x^n \right|^{\frac{1}{n}} = \left| e^{-n} x \right| < 1$$

$$\Rightarrow x < e^n \xrightarrow[n \to +\infty]{} +\infty$$

Donc $R = +\infty$

f)

Exercice 2

$\sum |a_n|$ diverge

$\Rightarrow \sum |a_n| 1^n$ diverge

donc $R \leq 1$

a_n tend vers 0 donc $\exists M$ tq $|a_n| \leq M$ $\forall n \in \mathbb{N}$

donc $|a_n| x^n \leq M x^n$

$M \sum x^n$ converge pour $|x| < 1$ au moins

donc $R \geq 1$

$\Rightarrow R = 1$

Exercice 3

i.e $\quad \sum a_n x^n \quad$ cv $\quad \forall |x| \leq R_1$

$\sum a_n x^{2n} \quad$ cv $\quad \forall |x|^2 \leq R_1$

$\Rightarrow |x| \leq \sqrt{R_1} = R_2$

Exercice 4

$$f(x) := \sum_{n \geq 0} \frac{x^{4n+1}}{4n+1}$$

1] $\left| \frac{1}{4n+1+4} x^{\cancel{4n+1}} x^4 \quad \frac{4n+1}{\cancel{x^{4n+1}}} \right|$

$\Rightarrow x^4 < \frac{4n+1}{4n+1} + \frac{4}{4n+1} \longrightarrow 1$

$x < 1$

$\Rightarrow R = 1$

2] D'après le cours, série entière converge normalement.

Donc $\left(\sum u_n x^n\right)' = \sum (u_n x^n)'$

Donc $\sum_{n \geq 0} \left(\frac{x^{4n+1}}{4n+1}\right)' = \sum (x^4)^n = \frac{1}{1-x^4}$

3] $f(x) = \int_0^x \frac{1}{1-t^4} \, dt = -\frac{1}{4} \ln(1-x^4)$

$$\frac{1}{(1-t^2)(1+t^2)} = \frac{a}{1-t^2} + \frac{b}{1+t^2}$$

$a1 + at^2 + b - bt^2$

$= a + b = 1 \quad \Rightarrow a = \frac{1}{2}$

$a - b = 0 \quad\quad b = \frac{1}{2}$

$$\int_0^x \frac{1}{1-t^4} dt = \int_0^x \frac{1}{2}\left(\frac{1}{1-t^2}\right) + \frac{1}{2}\left(\frac{1}{1+t^2}\right) dt$$

$$\int_0^x \frac{1}{1-t^4}\,dt = \int_0^x \frac{1}{2}\left(\frac{1}{1-t^2}\right) + \frac{1}{2}\left(\frac{1}{1+t^2}\right) dt$$

$$= \int_0^x \frac{1}{2}\left(\frac{1}{1+t^2}\right)dt + \frac{1}{2}\int_0^x \frac{1}{1-t^2}$$

$$= \frac{1}{2}\left[\arctan(t)\right]_0^x + \frac{1}{2}\int_0^x \frac{1}{(1-t)(1+t)}$$

$$\int_0^x \frac{1}{(1-t)(1+t)}\,dt = \int_0^x \frac{1}{2}\,\frac{1}{1-t} + \frac{1}{2}\,\frac{1}{1+t}\,dt$$

$$\left[-\frac{1}{2}\ln|1-t| + \frac{1}{2}\ln(1+t)\right]_0^x$$

$$= \frac{1}{2}\arctan(x) + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

Exercice 5

$$a(x) = \sum \frac{1}{n3^n}x^n$$

$$\left|\frac{\cancel{x^n}x}{(n+1)\cancel{3^n}3}\;\frac{n\cancel{3^n}}{\cancel{x^n}}\right| = \left|x\;\frac{n}{n+1}\,\frac{1}{3}\right| < 1$$

$$\Rightarrow x < 3\,\frac{n+1}{n} \longrightarrow 3$$

$$R = 3$$

$$a'(x) = \sum_{n\geq 1} \frac{1}{3^n}x^{n-1}$$

$$= \sum_{n\geq 0} \frac{1}{3^{n+1}}x^n = \frac{1}{3}\sum_{n\geq 0}\frac{1}{3^n}x^n = \sum\left(\frac{x}{3}\right)^n = \frac{1}{3}\,\frac{1}{1-\frac{x}{3}}$$

$$= \frac{1}{3}\,\frac{3}{3-x}$$

$$a(x) = \int_0^x a'(t)\,dt = \int_0^x \frac{1}{1-\frac{t}{3}}\,dt = -\ln\left(1-\frac{x}{3}\right)$$

$$f(x) = \sum_{n\geq 0} \frac{n+2}{n+1} x^n$$

$$\left| \frac{n+2+1}{n+2} \cancel{x^n} x \quad \frac{n+1}{n+2\,\cancel{x^n}} \right| = \left| x \frac{(n+3)(n+1)}{(n+2)(n+2)} \right| < 1$$

$$\Rightarrow x < \frac{(n+2)(n+2)}{(n+3)(n+1)}$$

$$= \frac{n^2+4n+4}{n^2+4n+3} \longrightarrow 1$$

Donc $R = 1$

$\forall x \in]-1,1[\quad f(x) = \sum_{n\geq 0} \frac{n+2}{n+1} x^n$

$$= \frac{1}{x} \sum_{n\geq 1} \frac{n+1}{n} x^n$$

$$= \frac{1}{x} \left(\sum_{n\geq 1} x^n + \sum_{n\geq 1} \frac{x^n}{n} \right)$$

$$= \frac{1}{x} \left(\frac{1}{1-x} - \frac{1-x}{1-x} - \ln(1-x) \right)$$

$$= \frac{1}{x} \left(\frac{x}{1-x} - \frac{\ln(1-x)}{x} \right)$$

$$= \begin{cases} \frac{1}{1-x} - \frac{\ln(1-x)}{x} & \forall x \in]-1,1[\setminus \{0\} \\ 2 & \text{si } x = 0 \end{cases}$$

c)

$$\sum_{n\geq 0} \frac{n-1}{n!} x^n$$

$$\left| \frac{n}{\cancel{n!}(n+1)} \cancel{x^n} x \quad \frac{\cancel{n!}}{\cancel{x^n}(n-1)} \right|$$

$$= x \frac{n}{(n+1)(n-1)} < 1$$

$$\Rightarrow x < \frac{n^2-1}{n} \xrightarrow[n \to +\infty]{} +\infty$$

Donc $R = +\infty$

$\forall x \in \mathbb{R} \quad C(x) = \sum_{n\geq 0} \frac{n-1}{n!} x^n$

$$= \sum_{n\geq 0} \frac{1}{(n-1)!} x^n - \sum \frac{x^n}{n!}$$

$$= x \sum_{n\geq 1} \frac{x^{n-1}}{(n-1)!} - \sum \frac{x^n}{n!}$$

$$= x \sum \frac{x^n}{n!} - \sum_{n\geq 0} \frac{x^n}{n!}$$

$$= xe^x - e^x = (x-1)e^x$$

<u>Exercice 6</u>

$$f = \sum_{n\geq 1} \frac{(-1)^{n+1}}{n(2n+1)} x^{2n+1}$$

1) $\left| \frac{\cancel{x^{2n+1}} x^2}{(n+1)(2n+3)} \frac{n(2n+1)}{\cancel{x^{2n+1}}} \right|$

$$= x^2 \frac{2n^2+n}{2n^2+2n+3n+1}$$

$$= x^2 \frac{2+\frac{1}{n}}{2+\frac{5}{n}+\frac{1}{n^2}}$$

$$= x^2 < \frac{2+\frac{5}{n}+\frac{1}{n^2}}{2+\frac{1}{n}} \xrightarrow[n\to+\infty]{} 1$$

$\Rightarrow x^2 < 1$

$\Rightarrow x < 1$

Donc $R = 1$

De plus $\sum_{n\geq 1} f_n$ cv en -1 et 1 aussi

donc domaine de f est $[-1, 1]$

2] $\forall x \in]-1, 1[$ $\quad f'(x) = \sum_{n \geq 1} \frac{(-1)^{n+1}}{n} x^{2n} = \ln(1+x^2)$

3] $f(x) = \int_0^x \ln(1+t^2)\,dt = \left[t\ln(1+t^2) \right]_0^x - \int_0^x \frac{2t^2}{1+t^2}$

$u = \ln(1+t^2)$ $\qquad v = t$

$u' = \frac{2t}{1+t^2}$ $\qquad v' = 1$

$$2\int_0^x \frac{t^2}{1+t^2} = 2\int_0^x 1 - \frac{1}{1+t^2}\,dt$$

$$= 2\left[t - \arctan(t) \right]_0^x$$

$$= 2x - 2\arctan(x)$$

$\Rightarrow f(x) = x\ln(1+x^2) + 2\arctan(x) - 2x$

4] $\quad \left| \frac{(-1)^{n+1}}{n(2n+1)} x^{2n+1} \right| \leq \left| \frac{1}{2n^2+n} \right| \underset{\forall x \in [-1,1]}{\leq} \left| \frac{1}{n^2} \right|$

$\sum \frac{1}{n^2}$ converge car série de Reinmann avec $\alpha > 1$

donc

$\sum f_n$ conerge normalement sur $[-1, 1]$

5] Comme on a montrer que f cv. normalement sur $[-1, 1]$, on peut poser $x = 1$

et on obtient $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 1^-} \sum \frac{(-1)^{n+1}}{n(2n+1)} x^{2n+1}$

$$= \sum_{n \geq 1} \frac{(-1)^{n+1}}{n(2n+1)}$$

par (4) $\quad f(1) = 1\ln(2) + 2\arctan(1) - 2$

$$= \ln(2) + \sqrt{2} - 2$$