Exo 1

$\frac{1}{1+t^2} = 1 - t^2 + t^4 - t^6 + t^8 + \dots$

$\arctan(x) = \int_0^x \frac{1}{1+t^2}\,dt = \int_0^x 1 - t^2 + t^4 - t^6 \dots dt$

$= \int_0^x \sum_{n=0}^{+\infty} (-1)^n t^{2n} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$

$\frac{(-1)^{n+1}}{2(n+1)+1} \cdot \frac{2n+1}{(-1)^n} = (-1)\frac{2n+1}{2n+3} \to (-1)\,1 \qquad |L| = 1 \qquad R = \frac{1}{1} = 1$

Donc la série converge sur $]-1,1[$

$\sum \frac{(-1)^n}{2n+1} 1$ converge absolument, donc $\arctan(1) = \sum \frac{(-1)^n}{2n+1} 1^{2n+1}$

Exo 2

$\frac{1}{1+t} = 1 - t + t^2 - t^3 + t^4 \dots$

$\frac{1}{1+t} = \sum_{n=0}^{+\infty} (-1)^n t^n$

$\left|\frac{(-1)^n (-1)}{(-1)^n}\right| = |(-1)| = 1 = L \qquad \Rightarrow R = \frac{1}{L} = \frac{1}{1} = 1$

CV sur $]-1,1[$ diverge en $x = 1$

Donc valide sur $]-1,1[$

Or converge normalement, donc

$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{+\infty} (-1)^n t^n dt = \sum_{n=0}^{+\infty} \int_0^x (-1)^n t^n\,dt = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} t^{n+1}$

De plus $\ln(1+x) = \int_0^x \frac{1}{1+t}\,dt$ donc $= \sum_{n=0}^{+\infty} \frac{(-1)^n}{n+1} t^{n+1}$

Exercice 6.6.3

We suppose that f is infinitely differentiable on $I \ni 0$

(2) $f = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$

As we supposed it is infinitely diff.

then $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3$

$f'(0) = a_1$

$f''(x) = 2a_2 + 2\cdot 3 a_3 x + 3\cdot 4 a_4 x^2$

$f''(0) = 2a_2 \Rightarrow a_2 = \frac{f''(0)}{2}$

$f'''(x) = 2\cdot 3 a_3 + 2\cdot 3\cdot 4\, a_4 x$

$f'''(0) = 2\cdot 3 a_3 \Rightarrow a_3 = \frac{f'''(0)}{3!}$

As we continue, each successive coeff a_n

$a_n = \frac{f^{(n)}(0)}{n!}$

Exercice 6.6.5

$$S_N(x) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\left| \frac{(-1)^n (-1)}{(2n+3)!} \cdot \frac{(2n+1)!}{(-1)^n} \right| = \left| (-1) \frac{1}{(2n+2)(2n+3)!} \right| = L \xrightarrow[n\to+\infty]{} 0 \Rightarrow R = +\infty$$

$$\frac{2^{2n+1}}{(2n+1)!} \xrightarrow[n\to+\infty]{} 0 \quad \text{(same (-2))}$$

then $S_N(x)$ CV uniformly on $[-2,2]$

but for any $R \in \mathbb{R}$ $\frac{R^{2n+1}}{(2n+1)!} \xrightarrow[n\to+\infty]{} 0$

then $S_n(x)$ cv. uniformly for on $[-R, R]$ $\forall R \in \mathbb{R}_+$

Exercise 6.6.7

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n \qquad = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N + \dots$$

$$S_N(x) = \sum_{n=0}^{N} a_n x^n \qquad = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N$$

$$E_N(x) = \sum_{n=0}^{+\infty} a_n x^n - \sum_{n=0}^{N} a_n x^n = \sum_{n=N+1}^{+\infty} a_n x^n = a_{N+1} x^{N+1} + a_{N+2} x^{N+2} + \dots$$

$n \in \{0, \dots, N\}$

$$E_N^{(N)}(x) = N!\, a_{N+1} x + (N+1)!\, a_{n+2} x^2 + \dots$$

$$E_N^{(N)}(0) = 0$$

for $n \in \mathbb{N}$ x will still be with all the coeffs that will give us 0 in $E_N^{(n)}(0)$