Analyse TD

Exercice 12

$$v_n = \int_{n\pi}^{(n+1)\pi} \frac{\sin t}{t}\,dt = \left[-\frac{\cos(t)}{t}\right]_{n\pi}^{(n+1)\pi} - \int_{n\pi}^{(n+1)\pi} \frac{\cos(t)}{t^2}\,dt$$

$u = \frac{1}{t} \quad u' = \frac{1}{t^2}$

$$v_n = \frac{\cos(n\pi)}{n\pi} - \frac{\cos((n+1)\pi)}{(n+1)\pi} - \int_{n\pi}^{(n+1)\pi} \frac{\cos t}{t^2}\,dt$$

$$= \frac{(-1)^n}{n\pi} + \frac{(-1)^n}{(n+1)\pi} - \int_{n\pi}^{(n+1)\pi} \frac{\cos(t)}{t^2}\,dt$$

$$v_n = (-1)^n\left(\frac{1}{n\pi} + \frac{1}{(n+1)\pi}\right) - \int_0^{\pi} \frac{\cos(t+n\pi)}{(t+n\pi)^2}\,dt$$

$$= (-1)^n \underbrace{\left(\frac{1}{n\pi} + \frac{1}{(n+1)\pi} + \int_0^{\pi} -\frac{\cos t}{(t+n\pi)^2}\,dt\right)}_{:= a_n}$$

i) $a_n > 0$ $\quad \frac{-\cos(t)}{t^2} \geq -\frac{1}{t^2}$

d'où $\int_0^{\pi} -\frac{\cos(t)}{(t+n\pi)^2}\,dt \geq \int_0^{\pi} -\frac{1}{(t+n\pi)^2}\,dt$

$$\int_0^{\pi} \frac{-\cos(t)}{(t+n\pi)^2}\,dt \geq \left[\frac{1}{t+n\pi}\right]_0^{\pi}$$

$$\frac{1}{(n+1)\pi} - \frac{1}{n\pi}$$

$a_n \geq \frac{2}{(n+1)\pi} > 0 \quad \forall n \in \mathbb{N}$

ii) $a_n \searrow 0$:

$\cancel{\int_0^{\pi} \frac{\cos t}{(t+n\pi)^2}\,dt \ \left[\frac{1}{t+n\pi}\right]_0^{\pi}}$

$\frac{2}{n\pi} \geq a_n \geq \frac{2}{(n+1)\pi} \geq a_{n+1} \geq \frac{2}{(n+2)\pi}$ (par encadrement du cos)

U_n décroissante On applique le critère des séries alternées

Exercice 13

$$v_n = \frac{(-1)^n}{n^\alpha + (-1)^n} = \frac{(-1)^n}{n^\alpha}\left(\frac{1}{1+\frac{(-1)^n}{n^\alpha}}\right)$$

$$= \frac{(-1)^n}{n^\alpha}\left(1 - \frac{(-1)^n}{n^\alpha} + o\left(\frac{1}{n^\alpha}\right)\right)$$

$$= \frac{(-1)^n}{n^\alpha} - \frac{1}{n^{2\alpha}} + o\left(\frac{1}{n^{2\alpha}}\right)$$

$\alpha < 0$ ~~car~~ diverge car $v_n \not\to 0$

$\alpha > \frac{1}{2}$ $\sum v_n$ ~~cv~~ converge car $-\frac{1}{n^{2\alpha}}$ cv

$\alpha \leq \frac{1}{2}$ $\sum v_n$ diverge car $\sum \frac{1}{n^{2\alpha}}$ div.

Exercice 50

1/a) $$\frac{1}{a+1} \leq \int_a^{a+1} \frac{1}{x}\,dx \leq \frac{1}{a}$$

b) $H_n \leq \ln(n)+1$ -> Chasles

$$\int_1^{n+1} \frac{1}{x}\,dx \leq \sum^{n} \frac{1}{k} \leq \int_1^{n} \frac{1}{x}\,dx + 1$$