Two variable functions

Def: Two variable functions is a function $f: D \longrightarrow \mathbb{R}$ that takes two variables or vector of two points.
$D \subseteq \mathbb{R}^2$

We can plot this function in 3D using parametric $(x, y, f(x,y))$ As we alwa_s did it with single variable $(x, f(x))$

Continuity

$f(x,y)$ is continuous if $(D \subseteq \mathbb{R}^2)$
We note $x = (x,y)$
$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad$ st if $x \in D$ $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Where we calculate $|x - x_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2}$

Differentiability

Let's review single value functions.

$f(x)$ is diff. if $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = a$ (some point we call derivative)

$$\lim_{x \to x_0} \Bigg| \underbrace{\frac{f(x) - f(x_0)}{x - x_0}}_{\text{derivative}} - \underset{\text{derivative}}{a} \Bigg| = 0$$

Let's rewrite it as Taylor
the way to write a function as a line
using taylor

$$f(x) = f(x_0) + \underset{\substack{\shortparallel \\ f'(x) \Rightarrow \text{derivative}}}{M}(x - x_0) + \overset{o(x)}{R_0(x)} \quad \text{where} \quad \lim_{x \to x_0} \frac{R(x)}{|x - x_0|} = 0$$

$$\Leftrightarrow f(x) - f(x_0) - M(x - x_0) = R(x)$$

$$\Leftrightarrow \frac{f(x) - f(x_0)}{|x - x_0|} - M = \frac{R(x)}{|x - x_0|}$$

Now let's do the same with 2 variable functions.

Let $f(x,y)$ and $X=(x,y)$ be a function. It is diff. if

$\exists\, \delta > 0$ st $|X - X_0| < \delta \Rightarrow X \in \mathcal{D}$ and

$$f(x,y) = f(x_0,y_0) + \underbrace{M(x-x_0) + N(y-y_0)}_{\text{same Taylor but two variable.}} + R(x,y)$$

and $$\lim_{X \to X_0} \frac{R(x,y)}{|X - X_0|} = 0$$

Partial derivatives

Let $f : \mathcal{D} \to \mathbb{R}$, $\mathcal{D} \subseteq \mathbb{R}^2$

$(x,y) \mapsto f(x,y)$

We note partial derivatives:

$\frac{\partial f}{\partial x} = F'(x) = f'(x,b) = f_x$

i.e we treat y, or x as constant

$\frac{\partial f}{\partial y} = F'(y) = f'(a,y) = f_y$

and take derivative with respect to x, then to y as we'd do it with single variable function

Higher order derivatives

→ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$

→ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = (f_x)_y$

→ $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

→ $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = (f_y)_x$

↳ we treat a derivative in the parentheses as a new two variable function and take new partial derivatives of it.