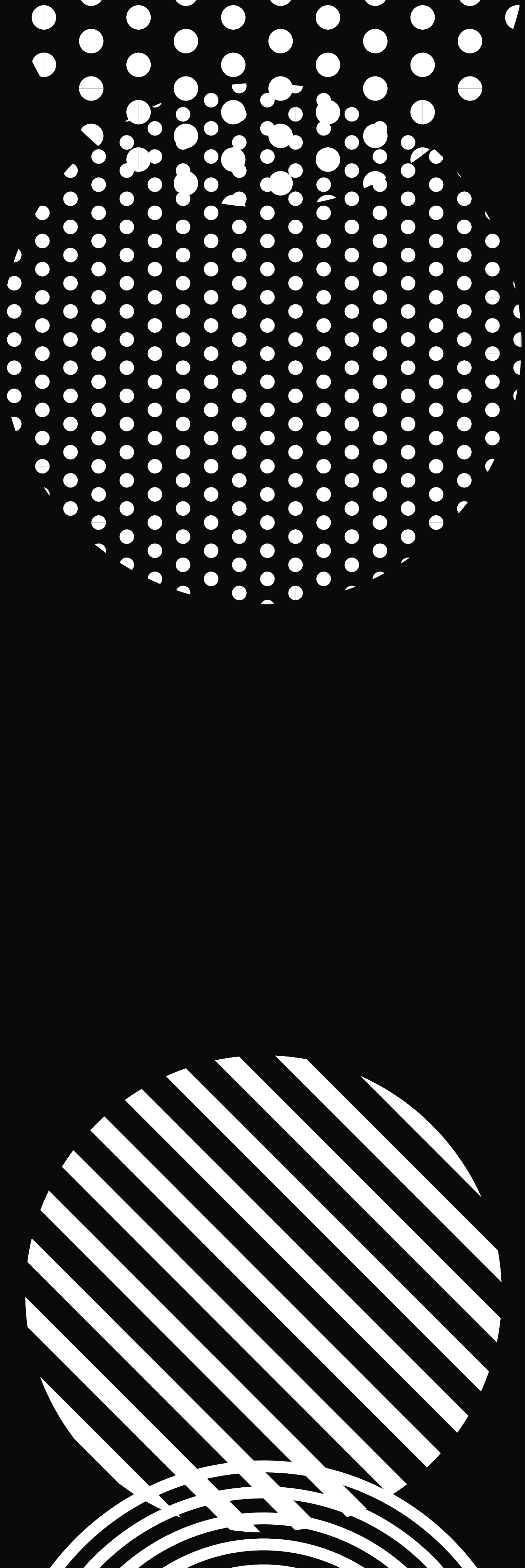

Exercice 7

Soit $(u_n)_{n\geq 1}$ décroissante, strict. pos.

1) $\sum_{k=n+1}^{2n} u_k \geq n u_{2n}$

$\forall k \in \{n+1, \dots, 2n\} \quad u_k \geq u_{2n}$

$\sum_{k=n+1}^{2n} u_k \geq \sum_{k=n+1}^{2n} u_{2n} = n u_{2n}$

2)

a) Par l'absurd:

Supposons que $n u_{2n} \not\longrightarrow 0$

D'après (1) $\sum_{k=n+1}^{2n} u_k \geq n u_{2n}$

de plus, par hyp $\sum u_n$ converge

donc $n u_{2n} \longrightarrow \ell \in \mathbb{R}_+$

$u_{2n} \sim \frac{\ell}{n}$

$\sum \frac{\ell}{n}$ diverge contrad

Donc $n u_n \xrightarrow[n\to+\infty]{} 0$

b) Contreexemple: Soit $u_n = \frac{1}{n\log(n)}$ $\quad n u_n = \frac{1}{\log(n)} \xrightarrow[n\to+\infty]{} 0$

Mais $\sum u_n$ diverge (Série de Bertrand)

3. On suppose que $\sum u_n$ converge, mq $\sum n(u_n - u_{n+1})$ aussi et a la même somme que $\sum u_n$

$\sum_{k=0}^{n} 0(u_0 - u_1) + (u_1 - u_2) + 2(u_2 - u_3) + 3(u_3 - u_4)$

$= 0 + u_1 + u_2 + u_3$

$\sum_{k=0}^{n} k(u_k - u_{k+1}) = \dots + k(u_k - u_{k+1}) + (k+1)(u_{k+1} - u_{k+2})$

$= \dots + k u_k - \cancel{k u_{k+1}} + \cancel{k u_{k+1}} + u_{k+1} - (k+1)u_{k+2}$

$$\sum_{k=1}^{n} k(u_k - u_{k+1}) = \sum_{k=1}^{n} k u_k - \sum_{k=1}^{n} k u_{k+1}$$

$$= \sum_{k=1}^{n} k u_k - \sum_{k=2}^{n+1} (k-1) u_k$$

$$= \sum_{k=1}^{n} k u_k - \sum_{k=2}^{n+1} (k-1) u_k$$

$$= u_1 + \sum_{k=2}^{n} (k u_k - k u_k + u_k) - n u_{n+1}$$

$$= u_1 + \sum_{k=2}^{n} u_k - n u_{n+1}$$

$$= \sum_{k=1}^{n} u_k - n u_{n+1}$$

Exo 8

Soit $\alpha > 0$ et $\beta > \frac{1}{2}$

$$u_n = \frac{1+n^5}{n^2+3^n}$$

$n+1 \underset{n \to +\infty}{\sim} n$

$$\frac{u_{n+1}}{u_n} = \frac{n^2+3^n}{(n+1)^2+3^{n+1}} \cdot \frac{1+(n+1)^5}{1+n^5} \quad \sim 1$$

$$\sim \frac{3^n}{3^{n+1}} \sim \frac{1}{3}$$

$$u_n = \left(\frac{2n-1}{n+1}\right)^{2n} \alpha^n$$

$$u_n^{\frac{1}{n}} = \left(\frac{2n-1}{n+1}\right)^2 \alpha$$

$$= \left(\frac{2-\frac{1}{n}}{1+\frac{1}{n}}\right)^2 \alpha \longrightarrow 4\alpha$$

Si $\alpha = \frac{1}{4}$ $\quad \alpha^n = \frac{1}{2^{2n}}$

$$u_n = \left(\frac{2n-1}{2(n+1)}\right)^{2n} = \left(\frac{1-\frac{1}{2n}}{1+\frac{1}{n}}\right)^{2n} \longrightarrow \frac{e^{-1}}{e^2} = e^{-3} \neq 0$$

Donc $\sum u_n$ diverge

$$u_n = \frac{n!}{n^n}$$

$$\frac{u_{n+1}}{u_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{\cancel{n!} \cdot (n+1) \cdot n^n}{(n+1)^{n+1} \cdot \cancel{n!}} = \frac{n^{n+1} + n^n}{(n+1)^{n+1}} = \left(\frac{n}{n+1}\right)^n$$

$$= \left(\frac{1}{1+\frac{1}{n}}\right)^n = \left(1+\frac{1}{n}\right)^{-n} \xrightarrow[n \to +\infty]{} e^{-1}$$

donc la série converge

d) $\left(\frac{n}{n+1}\right)^{\frac{1}{3}} - \left(\frac{n}{n+\alpha}\right)^{\frac{1}{2}}$

$$u_n = \left(1+\frac{1}{n}\right)^{-\frac{1}{3}} - \left(1+\frac{\alpha}{n}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{n}\left(-\frac{1}{3} + \frac{\alpha}{2}\right) + \frac{1}{n^2}\left(\frac{2}{\alpha} + \frac{\alpha^{23}}{8}\right) + o\left(\frac{1}{n^2}\right)$$

Si $\frac{\alpha}{2} = \frac{1}{3} \iff \alpha = \frac{2}{3}$ alors u_n converge

sinon $u_n \sim \frac{1}{n}$ donc diverge.

e) $\frac{1}{n^{1+1/n}}$

$$u_n = n^{-(1+1/n)}$$

$$= \exp\left(-\left(1+\frac{1}{n}\right)\log(n)\right)$$

$$= \exp\left(-\log(n) + \frac{\log(n)}{n}\right) \qquad \frac{\log(n)}{n} \xrightarrow[n\to+\infty]{} 0$$

d'où $u_n \sim \exp(-\log(n)) = \frac{1}{n}$

et Σu_n diverge.

f) $u_n = \frac{(n!)^2}{(2n)!}$

$$\frac{u_{n+1}}{u_n} = \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n!)}{(n!)^2}$$

$$= \frac{((n+1)!)^2}{(n!)^2 \quad (2n+1)(2n+2)} = \frac{(n+1)^2}{(2n+1)(n+1)\cdot 2} = \frac{n+1}{2(2n+1)} = \frac{1+\frac{1}{n}}{4+\frac{2}{n}} \xrightarrow[n\to+\infty]{} \frac{1}{4}$$

donc u_n converge

$$u_n = 1 - n \ln\left(\frac{2n+1}{2n-1}\right)$$

$$= 1 - n\left(\ln\left(1+\frac{1}{2n}\right) - \ln\left(1-\frac{1}{2n}\right)\right)$$

$$= 1 - n\left(\frac{1}{2n} - \cancel{\frac{1}{8n^2}} + \frac{1}{3.8n^3} - \left(-\frac{1}{2n} - \cancel{\frac{1}{8n^2}} - \frac{1}{3.8n^3}\right) + o\left(\frac{1}{n^3}\right)\right)$$

$$= 1 - 1 - \frac{1}{17n^2} + o\left(\frac{1}{n^2}\right)$$

$$= -\frac{1}{12n^2} + o\left(\frac{1}{n^2}\right)$$

h) $\frac{\ln^2(n)}{n^{6/5}+1} \sim \frac{1}{\ln^{-2}(n)\, n^{6/5}}$

qui est une série de Bertrand convergente

car $\frac{6}{5} > 1$.

i) $u_n = 1 - (\cos(\frac{1}{n^\beta}))^n$

$= 1 - \left(1 - \frac{1}{2n^{2\beta}} + \frac{1}{n^{2\beta}}\right)^n$

$= 1 - \exp\left(n \log\left(1 - \frac{1}{2n^{2\beta}} + o\left(\frac{1}{n^{2\beta}}\right)\right)\right)$

$= 1 - \exp\left(-\frac{1}{2n^{2\beta-1}} + o\left(\frac{1}{n^{2\beta-1}}\right)\right)$

$= 1 - \left(1 - \frac{1}{2n^{2\beta-1}} + o\left(\frac{1}{n^{2\beta-1}}\right)\right) \sim \frac{1}{2n^{2\beta-1}}$

il faut que $\beta > 1$ pour converge.

Ex 9

$u_n = \exp(-(\ln(n))^\alpha)$

si $\alpha < 0$ diverge ($u_n \not\to 0$)

si $\alpha = 1$