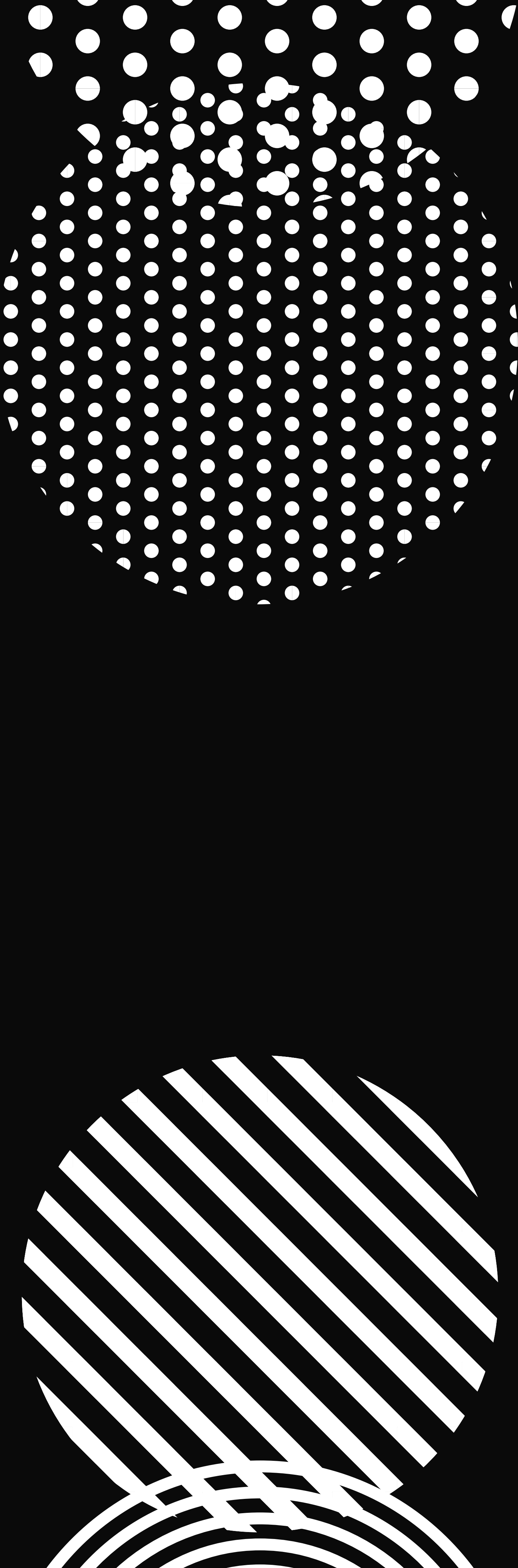

Exercice 1

$f(x) = 1 + 2x^2 - x^3 + o(x^4)$

$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + o(x^4)$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + o(x^2)$

$\ln(1+x^2) = x^2 - \frac{x^4}{2} + o(x^4)$

$1 - \cos(x) = 1 - 1 + \frac{x^2}{2} - \frac{x^4}{24} + o(x^4)$
$=$

$(1-\cos(x))\ln(1+x^2) = \left(\frac{x^2}{2} - \frac{x^4}{24}\right)\left(x^2 - \frac{x^4}{2}\right) + o(x^4)$
$= \frac{x^4}{2} - \frac{x^4}{4} + o(x^4)$
$= x^4\left(\frac{1}{2} - \frac{1}{4}\right) + o(x^4)$
$= x^4\left(\frac{2}{4} - \frac{1}{4}\right) + o(x^4)$
$= \frac{x^4}{4} + o(x^4)$

$h(x) = \frac{\sin(x^2)}{e^{x^2} - 1}$

$\sin(x) = x - \frac{x^3}{6} + o(x^4)$
$\sin(x^2) = x^2 - \frac{x^6}{6} + o(x^6)$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + o(x^4)$

$e^{x^2} = 1 + \frac{x^4}{2} + o(x^4$
$e^{x^2} - 1 = x^2 + \frac{x^4}{2} + o(x^4)$

$\frac{1}{e^{x^2}-1} = \left(e^{x^2} - 1\right)^{-1} = \left(x^2 + \frac{x^4}{2} + o(x^4)\right)^{-1}$

$$\left(x^2+\frac{x^4}{2}+o(x^4)\right)^{-1} = \left(x^2\left(1+\frac{x^2}{2}+o(x^2)\right)\right)^{-1}$$

$$=\left(x^2\right)^{-1}\cdot\left(1+\left(\frac{x^2}{2}+o(x^2)\right)\right)^{-1}$$

$$(1+x)^{-1} = 1+(-1)x+\frac{-1(-1-1)}{2}x^2+\frac{-1(-2)(-3)}{6}x^3+x^4\frac{-1(-2)(-3)(-4)}{24}+o(x^4)$$

$$= 1-x+x^2-x^3+x^4+o(x^4)$$

$$\left(1+\left(\frac{x^2}{2}\right)\right)^{-1} = 1-\frac{x^2}{2}+\frac{x^4}{4}+o(x^4) - \frac{x^6}{8}$$

$$\left(1+\frac{x^2}{2}\right)^{-1}\cdot x^{-2} = \frac{1}{x^2}-\frac{1}{2}+\frac{x^2}{4}-\frac{x^4}{8}+o(x^4)$$

$$\frac{\sin(x^2)}{e^{x^2}-1} = \left(x^2-\frac{x^6}{6}+o(x^6)\right)\left(\frac{1}{x^2}-\frac{1}{2}+\frac{x^2}{4}-\frac{x^4}{8}+o(x^4)\right)$$

$$= \underline{\underline{1-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^4}{6}+o(x^4)}}$$

Exercice 2

1) $\sin(x) = x + o(x)$

Donc $\sin(x) \underset{x\to 0}{\sim} x$

$\neq \pi x^3$

$\sin(x) = x - \frac{1}{6}x^3 + o(x^3)$

Donc $\sin(x) \underset{x\to 0}{\not\sim} x + \pi x^3$

2) $1 - \cos(x) = \frac{x^2}{2} - \frac{x^4}{24} + o(x^5)$

$$2x + \sin(x) = 2x + x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$= 3x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\tan(x) = x + \frac{x^3}{3} + \frac{2}{15}x^5 + o(x^5)$$

$$\tan(x^3) = x^3 + o(x^5)$$

$$x^4 + \tan(x^3) = x^3 + x^4 + o(x^5)$$

$$(1 - \cos(x))(2x + \sin(x)) = \left(\frac{x^2}{2} - \frac{x^4}{24}\right)\left(3x - \frac{x^3}{6} + \frac{x^5}{120}\right) + o(x^5)$$

$$= \frac{3x^3}{2} - \frac{3x^5}{24} + o(x^5)$$

$$\frac{\frac{3x^3}{2} - \frac{3x^5}{24}}{x^3 + x^4} = \frac{\frac{3x^3}{2} - \frac{3x^5}{24}}{x^3(1+x)} = \frac{\frac{3}{2} - \frac{3x^2}{24}}{1+x}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + o(x^4)$$

$$\left(\frac{3}{2} - \frac{3x^2}{24}\right)\left(1 - x + x^2 - x^3 + x^4\right) + o(x^4)$$

$$= \frac{3}{2} - \frac{x^2}{8} - \frac{3}{2}x + \frac{x^3}{8} - \frac{3x^3}{2} + \frac{3x^4}{2} + o(x^4)$$

$$\lim_{x \to 0} \frac{(1 - \cos(x))(2x + \sin(x))}{x^4 + \tan(x^3)} = \frac{3}{2}$$

3)

$$\frac{\arctan(x^\alpha)}{1+x^\alpha}$$

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)$$
$$\arctan(x^\alpha) = x^\alpha - \frac{x^{3\alpha}}{3} + \frac{x^{5\alpha}}{5} + o(x^{5\alpha})$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + o(x^5)$$

$$\frac{1}{1+x^\alpha} = 1 - x^\alpha + x^{2\alpha} - x^{3\alpha} + x^{4\alpha} - x^{5\alpha} + o(x^{5\alpha})$$

$$f(x) = \frac{\arctan(x^\alpha)}{1+x^\alpha} = \left(x^\alpha - \frac{x^{3\alpha}}{3}\right)\left(1 - x^\alpha + x^{2\alpha} - x^{3\alpha}\right)$$

$$= x^\alpha - x^{2\alpha} + x^{3\alpha} - \frac{x^{3\alpha}}{3}$$

si $\alpha > 0$

$$= x^\alpha - x^{2\alpha} + \frac{2x^{3\alpha}}{3} + o(x^{3\alpha})$$

Donc $f(x) \sim x^\alpha$

si $\alpha < 0$

$x^\alpha \xrightarrow[x \to 0]{} +\infty$ donc $\arctan(x^\alpha) = \frac{\pi}{2}$

$1 + x^\alpha \sim x^\alpha$

$f(x) = \frac{\pi}{2x^\alpha}$

Exercice 3

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

$$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{1}{x} \cdot \left(\frac{1}{1 - \frac{x}{2} + \frac{x^2}{3}} - 1 \right)$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + o(x^3)$$

$$\frac{1}{1 - \frac{x}{2} + \frac{x^2}{3}} = 1 + \frac{x^2}{2} - \frac{x^2}{3} + \frac{x^2}{4} - \frac{x^3}{3} + o(x^3)$$

$$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{1}{x}\left(\frac{x^2}{2} - \frac{x^2}{3} + \frac{x^2}{4} - \frac{x^3}{3}\right)$$

$$= \frac{x}{2} - \frac{x}{3} + \frac{x}{4} - \frac{x^2}{3} + o(x^2) = \frac{5}{12}x + o(x)$$

$$\frac{6}{12} - \frac{4}{12} + \frac{3}{12} = \frac{5}{12}$$

$$\frac{1}{\ln(1+x)} - \frac{1}{x} \underset{x\to0}{\sim} \frac{5}{12}x$$

$$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{x - \ln(1+x)}{x\ln(x+x)} = \frac{x - x + \frac{x^2}{2} + o(x^2)}{x\ln(1+x)}$$

$$= \frac{\frac{1}{2}x^2 + o(x^2)}{x\ln(1+x)} = \frac{1}{2}\,\frac{x}{\underset{x\to0}{\ln(1+x)}} \sim \frac{1}{2}$$

$$\ln(1+x) \underset{x\to0}{\sim} x$$

$$\frac{1}{\ln(1+x)} - \frac{1}{x} = \frac{1}{\ln(1+x)}\left(1 - \underbrace{\frac{\ln(1+x)}{x}}_{x\to+\infty\ \to 0}\right)$$

$$= \frac{1}{\ln(1+x)} \underset{x\to+\infty}{\sim} \frac{1}{\ln(x)}$$

$$\ln(x + \sqrt{1+x^2}) = \ln\left(x\left(1 + \frac{\sqrt{1+x^2}}{x}\right)\right) = \ln(x) + \ln\left(1 + \frac{\sqrt{1+x^2}}{x}\right)$$

$$\underset{x\to+\infty}{\sim} \ln(x) \qquad \sim \ln(2)$$

2) $f(x) = (x^3 + x^2)^{\frac{1}{3}} - \sqrt{x^2 + ax}$

$$= x\left(1 + \frac{1}{x}\right)^{\frac{1}{3}} - x\sqrt{1 + \frac{a}{x}}$$

$$= x\left(\left(1 + \frac{1}{x}\right)^{\frac{1}{3}} - \left(1 + \frac{a}{x}\right)^{\frac{1}{2}}\right) = x\left(1 + \frac{1}{3x} - \frac{\frac{2}{9}}{2x^2} - 1 - \frac{a}{2x} + \frac{\frac{1}{4}a^2}{2x^2} + o\left(\frac{1}{x^2}\right)\right)$$

$(1+x)^{\alpha}$

$$= x + \frac{1}{3} - \frac{2}{9}\cdot\frac{1}{2x} - \cancel{x} - \frac{a}{2} + \frac{1a^2}{8x}$$

$\cancel{x} + \frac{1}{3} - \frac{2}{9} \cdot \frac{1}{2x} \cancel{-x} - \frac{a}{2} + \frac{1a^2}{8x} = \frac{1}{3} - \frac{a}{2} - \frac{2}{18x} + \frac{a^2}{8x} + o\left(\frac{1}{x}\right)$

$= \frac{2-3a}{6} + \dots$

$\alpha \neq \frac{2}{3} \quad f(x) \sim \frac{2-3a}{6}$

$a = \frac{2}{3} \quad f(x) \sim -\frac{1}{18x}$

<u>Exercice 4</u>

Soit $n \geq 1$ et $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$\frac{1}{n(n+1)} = \frac{1}{n^2+n} \leq \frac{1}{n^2}$ $\quad \frac{1}{n^2}$ série divergeante.

par comparaison des séries de termes positifs

$\sum \frac{1}{n(n+1)}$ aussi converge.

$\sum_{n=1}^{+\infty} \frac{1}{n(n+1)} = 1 - \frac{1}{n+1} \approx 1$

Exo 6

1) $\frac{1}{\alpha n+1} = \int_0^1 t^{\alpha n}\, dt$

$$\sum_{n=0}^{N} u_n = \int_0^1 \frac{dt}{1+t^\alpha} + (-1)^N \int_0^1 \frac{t^{\alpha(N+1)}}{1+t^\alpha}\, dt$$

$$\frac{(-1)^n}{\alpha n+1} = (-1)^n \int_0^1 \frac{t^{\alpha n}(1+t^\alpha)}{(1+t)^\alpha}\, dt$$

$$= (-1)^n \int_0^1 \frac{t^{\alpha n}}{1+t^\alpha} + (-1)^n \int_0^1 \frac{t^{\alpha(n+1)}}{1+t^\alpha}\, dt$$

$$\int_0^1 \frac{t^{\alpha(N+1)}}{1+t^\alpha}\, dt \leq \int_0^1 t^{\alpha(N+1)}\, dt$$

$t^{\alpha(N+1)} \xrightarrow[N \to +\infty]{} 0$ pour $t \in [0,1[$

2) on utilise la formule trouvé en 1
et $\int \frac{1}{t+1} = \ln(1+x)$

$\int \frac{1}{1+t^2} = \arctan(x)$