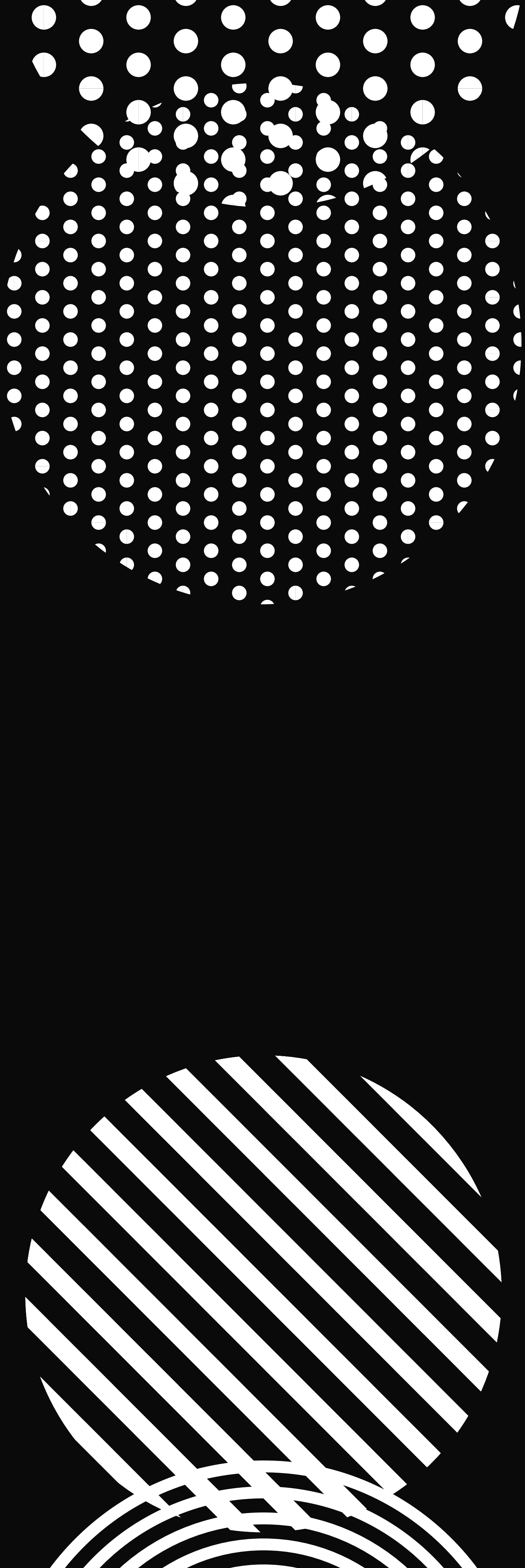

Exercice A

Soit $\lambda \in \mathbb{R}$ v.p $\quad \exists P \neq 0$

$\Phi(P) = \lambda P \quad$ i.e $\quad P(X+1) - P'(X) = \lambda P(X)$

$P = a_n X^n + \dots + a_1 X + a_0 \quad a_n \neq 0$

$P(X+1) = a_n (X+1)^n + \dots + a_0$

$= a_n (X^n + n X^{n-1} + \dots + 1) + \dots + a_0$

$= a_n X^n + \underbrace{\dots + a_0}_{\text{termes de deg } \leq n-1}$

$P'(X) = n a_n X^{n-1}$

$\Phi(P)(X) = a_n X^n + \underbrace{\dots}_{\deg \leq n-1} \quad = \lambda a_n X_n + \underbrace{\dots}_{\deg \leq n-1}$

$\Rightarrow a_n = \lambda a_n \underset{a_n \neq 0}{\Rightarrow} \lambda = 1$ la seule valeur propre.

$\Phi(1) = 1 \quad$ donc $\quad Sp(\Phi) = \{1\}$

2) Soit $P \neq 0$ tq $\Phi(P) = P$

Supposons par l'absurde que $\deg(P) \geq 2$

$P(X) = a_n X^n + a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \underbrace{O(X^{n-3})}_{\deg \leq n-3}$

$P(X+1) = a_n (X+1)^n + a_{n-1} (X+1)^{n-1} + a_{n-2} (X+1)^{n-2} + O(X^{n-3})$

$= a_n (X^n + n X^{n-1} + \binom{n}{2} X^{n-2} + O(X^{n-3}))$

$+ a_{n-1} (X^{n-1} + (n-1) X^{n-2} + O(X^{n-3}))$
$+ a_{n-2} X^{n-2} + O(X^{n-3})$

$= a_n X^n + (n a_n + a_{n-1}) X^{n-1} + (a_n \binom{n}{2} + a_{n-1}(n-1) + a_{n-2}) X^{n-2} + O(X^{n-3})$

$P'(X) = n a_n X^{n-1} + (n-1) a_{n-1} X^{n-2} + O(X^{n-3})$

$\Phi(X) = a_n X^n + a_{n-1} X^{n-1} + (a_n \binom{n}{2} + a_{n-2}) X^{n-2} + \dots$

$= P \quad = a_n X^n + a_{n-1} X^{n-1} + a_{n-2} X^{n-2} + \dots$

$\Rightarrow a_n \binom{n}{2} + a_{n-2} = a_{n-2} \Rightarrow a_n = 0$

donc $\deg(P) \leq 1$

donc est de la forme $P(X) = aX + b$

$P(X+1) - P'(X) = aX + a + b - a = P(X)$

donc $E_1(\Phi) = \mathbb{C}_1[X]$

$= \{ aX + b \mid a, b \in \mathbb{C} \}$

<u>Exercice B</u>

$\Phi(P) = XP'(X)$

Soit $\lambda \in \mathbb{R}$ v.p
donc $\exists P$ tq $XP'(X) = \lambda P(X)$

$P = a_n X^n + a_{n-1} X^{n-1} + \dots + a_1 X + a_0$
$P' = n a_n X^{n-1} + (n-1) a_{n-1} X^{n-2} + \dots + a_1$

$XP'(X) = n a_n X + (n-1) a_{n-1} X^{n-1} + \dots + a_1 X = \lambda a_n X^n + \lambda a_{n-1} X^{n-1} + \dots + \lambda a_1 X + \lambda a_0$

$\Rightarrow n a_n = \lambda a_n \Rightarrow n = \lambda$

$\lambda a_0 = 0 \Rightarrow \lambda = 0$ ou $a_0 = 0$

$Sp(\Phi) \subseteq \{0, 1, \dots, N\}$ $\quad \mathbb{C}_N[X]$

$P_n(X) = X^n$

$\Phi(P_n(X)) = n X^n = n P_n(X)$

$Sp(\Phi) = \{0, 1, \dots, N\}$

$E_n(\Phi) = \mathbb{R} X^n$

2) $\Phi \quad X \in \mathbb{R}_n[X] \longrightarrow (X-a) P'(X) \in \mathbb{R}_N[X]$

même raisonnement $Sp(\Phi) = \{0, \dots, N\}$

$E_n(\Phi) = \mathbb{R}(X-a)^n$

Exercice C

Soit $\lambda \in \mathbb{R}$, u tq

$Au = \lambda u$

$P(A) = B$

$Bu = P(A)u$

$= a_n A^n u + a_{n-1} A^{n-1} u + \dots + a_0$ val propre de B

$= (a_n \lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_0) u = \widetilde{P(\lambda)}\, u$

2) P est un polynome annulateur de A avec λ valeur propre de multiplicité dimension A

$P(A) = 0 \Rightarrow P(\lambda) = 0$

Exercice 2

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - Id$$

$$\left(\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - Id\right)^2 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}^2 - 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + Id^2$$

$$A^2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} = A + 2I_3$$

2) $$\begin{pmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{pmatrix} = \begin{vmatrix} -\lambda-1 & 1 & 1 \\ 1+\lambda & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 1+\lambda \begin{pmatrix} -1 & 1 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (1+\lambda)\begin{pmatrix} -1 & 0 & 1 \\ 1 & -\lambda-1 & 1 \\ 0 & 1+\lambda & -\lambda \end{pmatrix}$$

$$= (1+\lambda)^2 \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & -\lambda \end{pmatrix}$$

$$= (1+\lambda)^2 \begin{vmatrix} -1 & 0 & 0 \\ 1 & -1 & 2 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= (-1)(1+\lambda)^2 \begin{vmatrix} -1 & 2 \\ 1 & -\lambda \end{vmatrix}$$

$$= (-1)(1+\lambda)^2(\lambda - 2) = 0 \Rightarrow \lambda = -1 \text{ ou } \lambda = 2$$

$$\operatorname{Ker}\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} = \begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix} \Rightarrow \quad x=-y-z \quad \Rightarrow \operatorname{Vect}\begin{pmatrix}-y-z\\y\\z\end{pmatrix}$$

$$= \operatorname{Vect}\left(\begin{pmatrix}-1\\1\\0\end{pmatrix},\begin{pmatrix}-1\\0\\1\end{pmatrix}\right)$$

Donc A est diagonalisable.

3) $A^2 - A = 2I_3$

$\Rightarrow \frac{1}{2}A(A-Id) = I_3$

$\Rightarrow A\cdot\left(\frac{1}{2}(A-Id)\right) = I_3$

Donc $A^{-1} = \frac{1}{2}(A-Id)$

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2) $A^2 - A - 2I_3 = 0$

$X^2 - X - 2$ annule A

$X^2 - X - 2 = (X+1)(X-2)$ donc A est diagonalisable.

4) Si $n \in \mathbb{N}$, par division euclidienne, $\exists\, Q, R_n$ tq $\deg(R_n) \leq 1$

$$X^n = (X+1)(X-2)Q + R_n$$

donc $R_n = a_n X_n + b_n$

5) $X^n = (X+1)(X-2)Q + a_n X + b_n$

$(-1)^n = -a_n + b_n \qquad (X=-1)$

$2^n = 2a_n + b_n \qquad (X=2)$

$$\begin{cases} b_n = \dfrac{2^n + 2(-1)^n}{3} \\ a_n = \dfrac{2^n - (-1)^n}{3} \end{cases}$$

$$A^n = \underbrace{\cancel{(A+1)(A-2)}}_{\text{pol annulateur}} Q(A) + a_n A + b_n$$

$$A^n = a_n A + b_n = \begin{pmatrix} b_n & a_n & a_n \\ a_n & b_n & a_n \\ a_n & a_n & b_n \end{pmatrix}$$

Exercice I

1)
$$A = \begin{pmatrix} 3 & -2 & -2 \\ -4 & 1 & 2 \\ 8 & -4 & -5 \end{pmatrix}$$

$$\chi_A = \begin{vmatrix} 3-\lambda & -2 & -2 \\ -4 & 1-\lambda & 2 \\ 8 & -4 & -5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -2 & 0 \\ -4 & 1-\lambda & 1+\lambda \\ 8 & -4 & -\lambda-1 \end{vmatrix}$$

$$= 3-\lambda \begin{vmatrix} 1-\lambda & 1+\lambda \\ -4 & -(\lambda+1) \end{vmatrix} + 2 \begin{vmatrix} -4 & 1+\lambda \\ 8 & -(1+\lambda) \end{vmatrix}$$

$$= 3-\lambda[\lambda^2 - 1 + 4 + 4\lambda] + 2[4 + 4\lambda - 8 \quad -8\lambda)$$

$$= (3-\lambda)(\lambda^2 + 4\lambda + 3) - 8 - \lambda$$

$$= (3-\lambda)(\lambda^2 + 4\lambda + 3) - 8(\lambda+1)$$

$$\chi_A = (3-\lambda)(\lambda+1)(\lambda+3) - 8(\lambda+1)$$

$$= (\lambda+1)((3-\lambda)(\lambda+3) - 8) = (\lambda+1)(-\lambda^2 + 9 - 8)$$

$$= (\lambda+1)(\lambda+1)(\lambda-1)$$

$$\mathrm{Ker}\begin{pmatrix} 4 & -2 & -2 \\ -4 & 2 & 2 \\ 8 & -4 & -4 \end{pmatrix} = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \quad 4x = 2y + 2z \quad \Rightarrow \begin{pmatrix} \frac{1}{2}y + \frac{1}{2}z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$E_1 = \mathrm{Ker}\begin{pmatrix} 2 & -2 & -2 \\ -4 & 0 & 2 \\ 8 & -4 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$(S) \Leftrightarrow \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A\begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{où} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = e^{tA}\begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$$

$$\exp(A) = \sum_{n\geq 0} \frac{M^n}{n!} \qquad \exp(tA) = \exp(t\, PDP^{-1}) = \sum_{n\geq 0} \frac{(tPDP^{-1})^n}{n!}$$

$$= \sum_{n\geq 0} t^n \frac{(PDP^{-1})^n}{n!} = P\left(\sum_{n\geq 0} t^n \frac{D^n}{n!}\right) P^{-1} = P\left(\sum_{n\geq 0} \begin{pmatrix} \frac{t^n}{n!} & 0 & 0 \\ 0 & \frac{(-t)^n}{n!} & 0 \\ 0 & 0 & \frac{(-t)^n}{n!} \end{pmatrix}\right) P^{-1}$$

$$= P\left(\begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix}\right) P^{-1}$$

Donc $\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = P \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & e^{-t} \end{pmatrix} P^{-1} \begin{pmatrix} x(0) \\ y(0) \\ z(0) \end{pmatrix}$

Exercice I

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

$$\det A - \lambda I = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 9$$

$$= 1 - 2\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 2\lambda - 8$$

$$\Rightarrow \Delta = 4 - 4\cdot 1 \cdot (-8)$$

$$= 4 + 32 = 36$$

$$\lambda_1 = \frac{2-6}{2} = -2$$

$$\lambda_2 = \frac{2+6}{2} = 4$$

$$\chi_A = (\lambda+2)(\lambda-4)$$

$$E_{-2} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \Rightarrow x = -y \Rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$E_4 = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}$$

Soit $n \in \mathbb{N}$

$$A^n = PD^nP^{-1} = P\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^{2n} \end{pmatrix} P^{-1} \Rightarrow \begin{pmatrix} u_n \\ v_n \end{pmatrix} = P\begin{pmatrix} (-2)^n & 0 \\ 0 & 2^{2n} \end{pmatrix} P^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$P^{-1} = \left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} -1 & 1 & 1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & -1 & -1 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

$$A^n = \begin{pmatrix} -(-2)^n & 2^{2n} \\ (-2)^n & 2^{2n} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} (-2)^{n-1} + 2^{2n-1} & 2^{2n-1} - (-2)^{n-1} \\ 2^{2n-1} - (-2)^{n-1} & (-2)^{n-1} + 2^{2n-1} \end{pmatrix}$$

$$A^n \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} (-2)^{n-1} + 2^{2n-1} & 2^{2n-1} - (-2)^{n-1} \\ 2^{2n-1} - (-2)^{n-1} & (-2)^{n-1} + 2^{2n-1} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2^{2n} \\ 2^{2n} \end{pmatrix}$$

Thm : $A, B \in \mathcal{M}_n(\mathbb{R})$

$$\begin{cases} AB = BA \\ A, B \text{ diagonalisables} \end{cases}$$

$\exists$ P et D_1, D_2 diagonalisables

tq $\begin{cases} A = P D_1 P^{-1} \\ B = P D_2 P^{-1} \end{cases}$

Exercice R

E ev de dim. n sur $\mathbb{R}$ et $u, v \in End(E)$

1) $uv = vu$

a) Soit λ une v.p de u i.e $\exists x \in E$ tq

$u(x) = \lambda x$

$u(v(x)) = v(u(x)) = v(\lambda x) = \lambda v(x)$

donc $u(v(x)) = \lambda v(x)$ d'où $v(x)$ est un vecteur propre de valeur propre λ, d'où E_λ est stable par v.

b) Supposons que v est diagonalisable

donc $\exists \lambda_1, \dots, \lambda_p$ avec $E = E_{\lambda_1} \oplus \dots \oplus E_{\lambda_p}$

avec $\forall x \in E_{\lambda_i}$ $v(x) = \lambda x$

D'après a) $v(E_\lambda) \subset E_\lambda$

Or v est diagonalisable, donc $\forall i \in [\![1, \dots, p]\!]$, $v(E_{\lambda_i}) \subset E_{\lambda_i}$

donc $\exists j \in [\![1, p]\!]$ tq $E_\lambda \subset E_{\lambda_j}$, d'où $\forall x \in E_\lambda$

$v(x) = \lambda_j x$ d'où $v_{|E_\lambda}$ est diagonalisable

REFAIRE

c)

$E = \bigoplus_{\lambda \in Sp(u)} E_\lambda(u)$, v agit sur $E_\lambda(u)$ pour $\lambda \in Sp(u)$

Soit e_i^λ