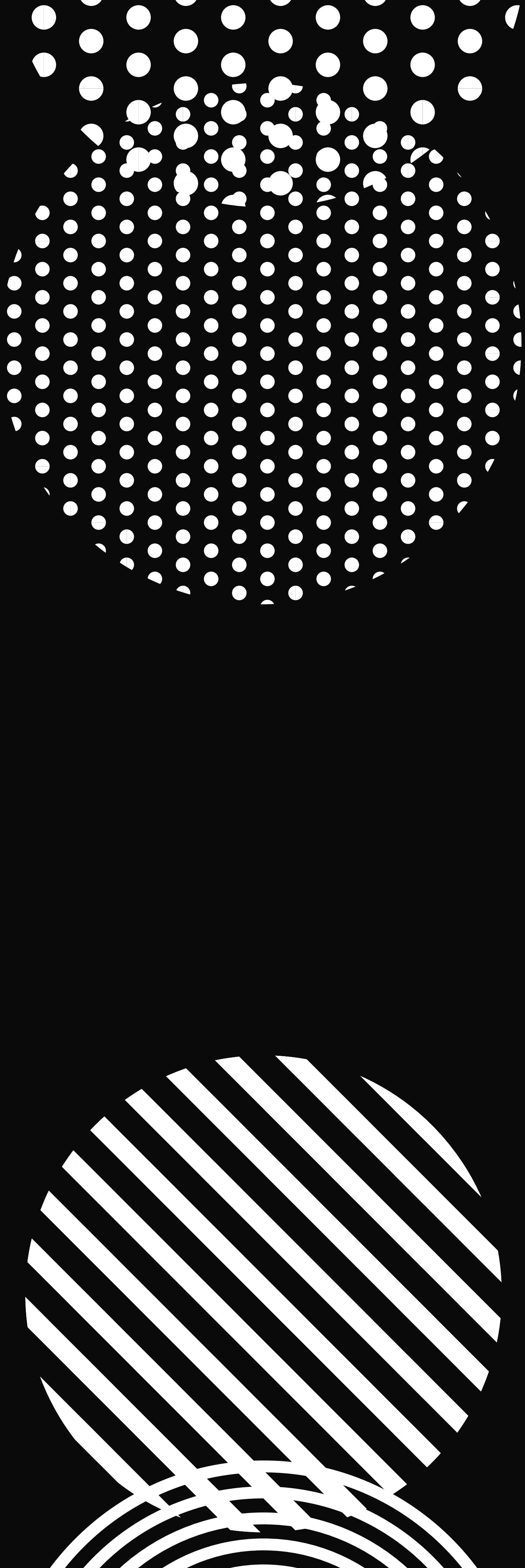

Exercice 1

(1) $A = {}^tA$

${}^tA = -A$

$\det(A\,{}^tA) = \det(A)\det({}^tA) = \det(I_n) = 1$

(2) Soit $A \in \mathcal{M}_n(\mathbb{R})$

Supposons que $A^2 = -I_n$

$\det(A^2) = \det(-I_n)$

$= \det(A)^2 = (-1)^n \Rightarrow n$ pair

(3) A antisymétrique de dim $2n+1$

${}^tA = -A$

$\det({}^tA) = \det(-A) = (-1)^{2n+1}\det(A)$

$\overset{\shortparallel}{\det(A)}$

$\det(A) = -\det(A) \Rightarrow \det(A) = 0$

c.ex.

Matrice de dim. 2

$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ $\det(A) = 1 \neq 0$

antisymétrique

Exercice 2

$$d := \det_B(v_1, v_2, v_3)$$

(1) $\det_B(v_1 + 3v_2,\ 2v_1 + v_2 - v_3,\ v_3 - v_1)$

$= |v_1, 2v_1 + v_2 - v_3, v_3 - v_1| + |3v_2, 2v_1 + v_2 - v_3, v_3 - v_1|$

$= 2\underbrace{|v_1, v_1, v_3 - v_1|}_{=0} + |v_1, v_2, v_3 - v_1| + \underbrace{|v_1, -v_3, v_3 - v_1|}_{=0}$

$+ 3|v_2, 2v_1 + v_2 - v_3, v_3 - v_1|$

$= d + 3|v_2, 2v_1 + v_2 - v_3, v_3 - v_1|$

$= d + 3\left(2|v_2, v_1, v_3 - v_1| + \underbrace{|v_2, v_2, v_3 - v_1|}_{=0} + |v_2, -v_3, v_3 - v_1|\right)$

$= d - 6|v_1, v_2, v_3| + 3\underbrace{|v_2, -v_3, v_3|}_{=0} + 3|v_2, -v_3, -v_1|$

$= d - 6d + 3d$ $\qquad -|-v_1, -v_3, v_2|$

$= d - 3d = \underline{-2d}$ $\qquad = |-v_1, v_2, -v_3|$

2) $\det(v_1 - v_3,\ v_2 + v_3,\ v_1 + v_3)$

$= 2\det(v_1, v_2 + v_3, v_1 + v_3)$

$= 2\det(v_1, v_2 + v_3, v_3)$

$= \underline{2d}$

$d(a, c + d) = d(a, c) + d(a, d)$

$d(a, \lambda b) = \lambda\, d(a, b)$

$d(a, a) = 0$

Exercice 3

Exercice 4

$B = -2\lambda$

$$\det(C) = \det({}^tC) = \begin{vmatrix} -\lambda & \lambda & 1 \\ \lambda & \lambda & 0 \\ 2\lambda^3 & 1 & 1 \end{vmatrix} = \lambda \begin{vmatrix} -1 & \lambda & 1 \\ \lambda & \lambda & 0 \\ 2\lambda^2 & 1 & 1 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} -1 & \lambda & 1 \\ \lambda & \lambda & 0 \\ 2\lambda^2+1 & 1-\lambda & 0 \end{vmatrix}$$

$$= \lambda \begin{vmatrix} \lambda & \lambda \\ 2\lambda^2+1 & 1-\lambda \end{vmatrix}$$

$$= \lambda(\lambda(1-\lambda) - \lambda(2\lambda^2+1))$$

$$= \lambda(\lambda - \lambda^2 - 2\lambda^3 - \lambda)$$

$$= -\lambda^3 - 2\lambda^4 = -\lambda^3(1+2\lambda)$$

$$\begin{matrix} 1+i & -1 & 2i & 1+i & -1 \\ i & 0 & 1 & i & 0 \\ 1 & i & 1 & 1 & 1 \end{matrix}$$

$$(1+i)(-i) + (i-1) + 2i(-1)$$

$$-i+1 + i - 1 - 2i = -2i$$

$$A = \begin{pmatrix} 1+i & -1 & 2i \\ i & 0 & 1 \\ 1 & i & 1 \end{pmatrix}$$

$$= (-1)(-1)\begin{vmatrix} i & 1 \\ 1 & 1 \end{vmatrix} - i \begin{vmatrix} 1+i & 2i \\ i & 1 \end{vmatrix}$$

$$= (i-1) - i((1+i) - (-2))$$

$$= i - 1 - i(3+i)$$

$$= i - 1 - 3i + 1 = -2i$$

Exercice 5

$$M = \begin{vmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2\pi}} & \ln(3) & 0 \\ & I & 1 \end{vmatrix} = - \begin{vmatrix} 1 & 0 & 0 \\ \ln(3) & 1 & 0 \\ \sqrt{2\pi} & \sqrt{2\pi} & 1 \end{vmatrix} = \underline{-1}$$

donc M inversible

$$K = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \frac{3}{3} & 0 \\ 0 & 1 & 2 \end{pmatrix} = \frac{3}{2} \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= \frac{3}{2} \times 2 - (-1)$$

$$= 3 + 1 = 4$$

Exercice 6

$$I_1 = \begin{vmatrix} a & a & a & a \\ a & b & b & b \\ a & b & c & c \\ a & b & c & d \end{vmatrix} = \begin{vmatrix} a & a & a & a \\ 0 & b-a & b-a & b-a \\ 0 & b-a & c-a & c-a \\ 0 & b-a & c-a & d-a \end{vmatrix} = a \begin{vmatrix} b-a & b-a & b-a \\ b-a & c-a & c-a \\ b-a & c-a & d-a \end{vmatrix}$$

$$= a \begin{vmatrix} b-a & b-a & b-a \\ 0 & c-b & c-b \\ 0 & c-b & d-b \end{vmatrix}$$

$$= a(b-a) \begin{vmatrix} c-b & c-b \\ c-b & d-b \end{vmatrix}$$

$$= a(b-a)(c-b)(d-c)$$

$$I_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix} \sim L_3 \leftarrow L_3 + L_2 \quad = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a+b+c & a+b+c & a+b+c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$J_4 = \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = abc \begin{vmatrix} b-a & c-a \\ b^2-a^2 & c^2-a^2 \end{vmatrix}$$

$$= abc\,(b-a)\,(c-a) \begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$$

$$= (abc)(b-a)(c-a)(c-b)$$

Exercice 8

(1) $\begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 3 \\ 4 & 6 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 2 & 3 \\ 4 & 3 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 4 & -1 & 6 \end{vmatrix}$

$= 2 \begin{vmatrix} 0 & 3 \\ -1 & 6 \end{vmatrix} = 6$

(2) $\begin{vmatrix} m & 1 & 1 \\ 1 & m & 1 \\ 1 & 1 & m \end{vmatrix} = m \begin{vmatrix} m & 1 \\ 1 & m \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 1 & m \end{vmatrix} + \begin{vmatrix} 1 & m \\ 1 & 1 \end{vmatrix}$

$= m(m^2-1) - (m-1) + (1-m)$

$= m^3 - m - m + 1 + 1 - m = m^3 - 3m + 2$

$= m(m^2-3) + 2$

$$\begin{matrix} 1 \\ x \\ x^2 \end{matrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$= m^3 - 2m - m + 2$

$= m^3 - m \quad -2m + 2 = m(m^2-1) + 2(-m+1)$

$\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 6 \\ 2 & 1 \end{vmatrix}$ (1)

$1 + 1 = 2$ in

Exercice 10

<u>n = 1</u> : $\det A = a_{1,1}$ et $0 \leq a_{1,1} \leq 1$

<u>$A \in \mathcal{M}_{n+1}(\mathbb{R})$</u> :

$$|\det A| = \left| \sum_{i=1}^{n+1} (-1)^{1+i} a_{1,i} \det(A_{1,i}) \right|$$

$$\leq \sum_{i=1}^{n+1} |a_{1,i} \det(A_{1,i})|$$

$$\leq \sum_{i=1}^{n+1} a_{1,i} \, |\det(A_{1,i})|$$

$$\leq \sum_{i=1}^{n+1} a_{1,i} \cdot 1 \qquad \text{par hyp de rec}$$

$$\leq 1$$