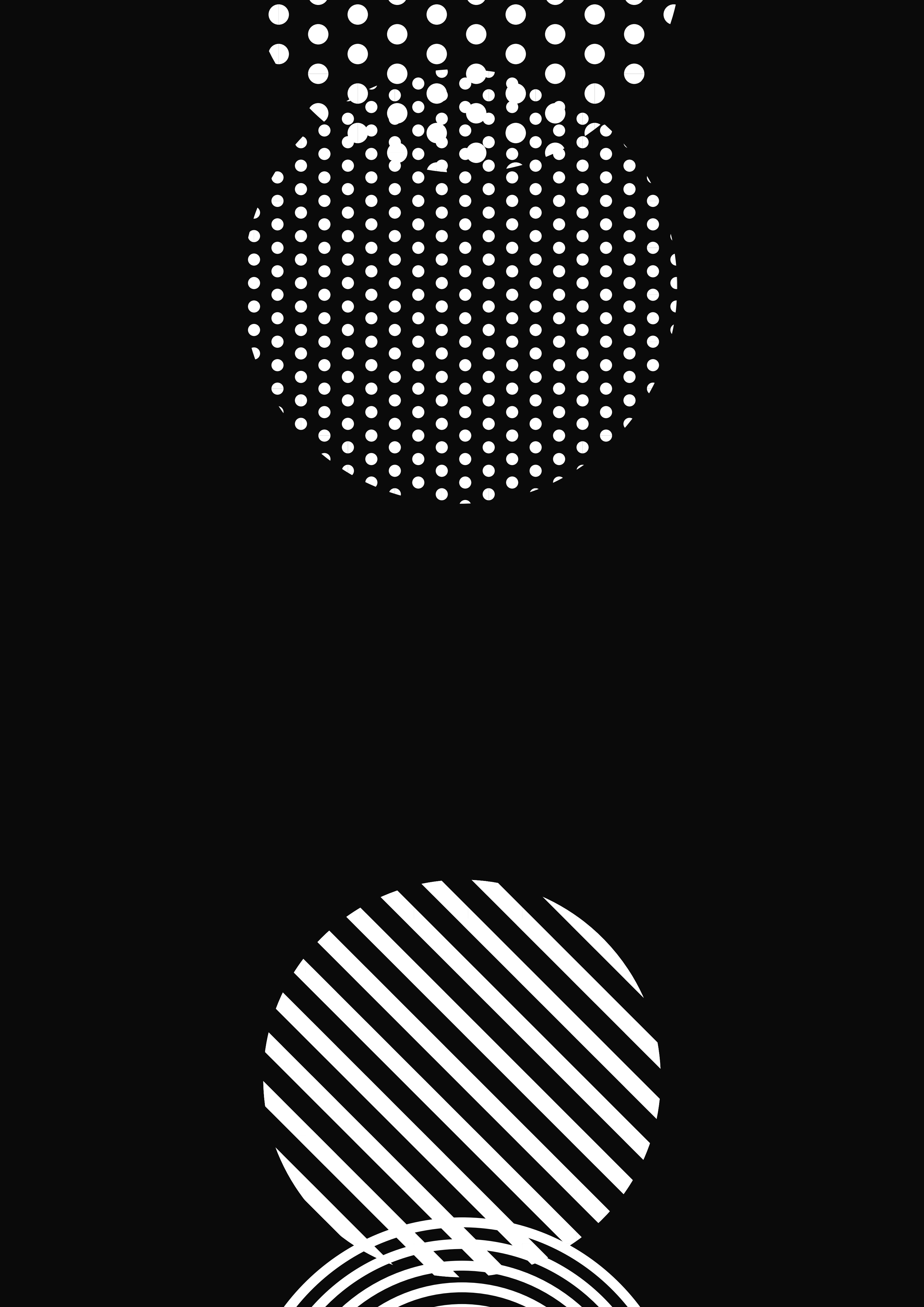

Exercice 1

1) $rg(A) = \dim(\mathrm{Im}(A)) = 2$

2) $\dim(\mathrm{Im}(A)) = 2$

Donc $c_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ et $c_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

Donc (c_1, c_2) est une base de $\mathrm{Im}\, f$

$\dim(\mathrm{Ker}\, f) = 1$

$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$ donc $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ est une base de $\mathrm{Ker}\, f$.

Exercice 2

1) $TR(AB) = \sum_{i=1}^{n} c_{i,i} = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,k} b_{k,i}$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad ||$

$Tr(BA) = \sum_{i=1}^{n} C_{i,i} = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{i,k} a_{k,j}$

A et B semblables => conjugué i.e $PAP^{-1} = B$

$\det(B) = 3 + 2\cdot(-1-2)$
$= 3 - 6 = -3$

$Tr(A) = 3$
$Tr(B) = 4$ *

$Tr((PA)P^{-1}) = Tr(P^{-1}(PA))$ par (1)
$= Tr(A) \neq Tr(B)$

2) $AB = (c_{i,j})$

$c_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$

si $i > j$:

$i < k$ et $j < k \Rightarrow b_{k,j} = 0$

$i > k \qquad\qquad a_{i,k} = 0$

$j < i = k \Rightarrow j < k \qquad b_{k,j} = 0$

$\vec{0} \in$

Exercice 3

Soit A et $B \in T_n$

$A = \begin{pmatrix} 0 & 0 & 0 \\ & \ddots & 0 \end{pmatrix}$ $\quad \forall i,j \quad a_{i,j} = 0 \Rightarrow i > j = a_{i,j} = 0$

$A+B = (a_{i,j} + b_{i,j})$ où $a_{i,j}$ et $b_{i,j} = 0$ pour $i > j$
Donc $A+B \in T_n$

$\lambda A = (\lambda a_{i,j})$ mais $a_{i,j} = 0$ pour $i > j$
donc $\lambda A \in T_n$
donc un espace vectoriel.

$\frac{n(n+1)}{2}$

2) a) $f(B \in T_n) = (A \in T_n \cdot B) \in T_n$ par 2.2

b) Supposons A inversible

$\mathrm{Ker}(f) = \{X \in T_n \mid AX = 0\}$

$AX = 0 \Rightarrow A^{-1}AX = A^{-1}0$
$\Rightarrow \underline{X = 0}$

Donc $\mathrm{Ker}(\tilde{f}) = \{\vec{0}\}$ et
f endomorphisme, donc

$\tilde{f}$ bijectif.

c) Comme $\tilde{f}$ est surjectif $\exists X \in T_n$ tq $f(X) = I_n$

$AX = I_n \Rightarrow X = A^{-1}$
donc $A^{-1} \in T_n$.

6, 7, 5 (4 et 8)

Exercice 6

1] $\mathbb{R}_3[X]$

$(1, X, X^2, X^3)$

$P = aX^3 + bX^2 + cX + d$

$P' = 3aX^2 + 2bX + c$

$3aX^3 + 2bX^2 + cX - 3aX^2 - 2bX - c + 2a(X+1)^3 + 2b(X+1)^2 + 2c(X+1) + 2d$

$(X+1)^3 = X^3 + 3X^2 + 3X + 1$

$(X+1)^2 = X^2 + 2X + 1$

$X^3(3a + 2a)$
$+ X^2(2b - 3a + 6a + 2b)$
$+ X(c - 2b + 6a + 4b + 2c)$
$+ 1(-c + 2a + 2b + 2c + 2d)$

$(5a, 3a + 4b, 6a + 2b + 3c, 2a + 2b + c + 2d)$

2] Oui,

1] Sol de prof

$f(1) = 2$
$f(X) = (X-1)2X + 2(X+1) = 3X + 1$
$f(X^2) = 4X^2 + 2X + 2$
$f(X^3) = 5X^3 + 3X^2 + 6X + 2$

$$\begin{array}{c} \\ 1 \\ X \\ X^2 \\ X^3 \end{array} \begin{array}{c} \begin{array}{cccc} f(1), & f(X), & f(X^2), & f(X^3) \end{array} \\ \begin{pmatrix} 2 & 1 & 2 & 2 \\ 0 & 3 & 2 & 6 \\ 0 & 0 & 4 & 3 \\ 0 & 0 & 0 & 5 \end{pmatrix} \end{array}$$

donc f bijective.

Exercice 7

$$\begin{pmatrix} 2 & 1 \\ 3 & -1 \\ -1 & -2 \end{pmatrix}$$

$2a + b = 3c + 5d$
$3a - b = 7c$
$-a - 2b = -7d$

$5a = 10c + 5d$

$\Rightarrow a = 2c + d$
$c + b = 3d$
$-2c - d - 2b = -7d \Rightarrow -2c - 2b = -6d$

$\Rightarrow$

$$a\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + b\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= \begin{cases} 2a + b = x \\ 3a - b = y \\ -a - 2b = z \end{cases}$$

$$\begin{cases} 3a + 5b = x \Rightarrow \\ 7a = y \Rightarrow a = \frac{y}{7} \\ -9b = z \Rightarrow b = -\frac{3}{7} \end{cases}$$

$= 5a = x + y \Rightarrow a = \frac{x+y}{5}$

$b = x - 2a \Rightarrow b = \frac{5x}{5} - 2\frac{x+y}{5}$

$= \frac{3x - 2y}{5}$

$5b = x - 3a$
$b = x - 3a\left(\frac{1}{5}\right)$
$b = \left(x - \frac{3y}{7}\right)\frac{1}{5}$
$= \frac{x}{5} - \frac{3y}{35}$
$= \frac{7x - 3y}{35} = b$

$a = \frac{y}{7}$

$\frac{x+y}{5} = \frac{y}{7}$

$\frac{3x - 2y}{5} = \frac{7x - 3y}{35}$

$\Rightarrow 3x - 2y = \frac{7x - 3y}{7}$

Exercice 2

Solution de prof

$\dim(E) = \dim(F) = 2$

$$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & -1 \\ 1 & -1 & -2 \end{pmatrix} = \vec{i}(-6-1) - \vec{j}(-4+1) + \vec{k}(-2-3)$$

$$\begin{vmatrix} \vec{i} & 2 & 1 \\ \vec{j} & 3 & -1 \\ \vec{k} & -1 & -2 \end{vmatrix} = \begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix}$$

$$\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 7 & 0 \\ 5 & 0 & -7 \end{pmatrix} = \begin{pmatrix} -49 \\ 21 \\ -35 \end{pmatrix} = 7\begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix}$$ donc il sont colinéaire.

$$\langle \vec{n}, \begin{pmatrix} -7 \\ 3 \\ -5 \end{pmatrix} \rangle = \vec{0} \Rightarrow \underline{-7x + 3y - 5z = 0}$$

Exercice 5

$\mathrm{Ker}(f) = \{x \mid f(x) = \vec{0}\}$

$\mathrm{Ker}(f^2) \subset \mathrm{Ker}(f)$

$\{x \mid f(f(x)) = \vec{0}\} = \mathrm{Ker}(f) + \{x \mid f(x) \in \mathrm{Ker}(f)\}$

Supposons $\{x \mid f(x) \in \mathrm{Ker}\, f\} = \vec{0}$

$f(x) \notin \mathrm{Ker}\, f \Rightarrow \mathrm{Ker}\, f \cap \mathrm{Im}\, f = \{\vec{0}\}$

$\forall x \in E \setminus \mathrm{Ker}\, f, \quad f(x) \in \mathrm{Im}\, f$

$\mathrm{Im}(f^2) \subset \mathrm{Im}(f)$

$\mathrm{Im}\, f \subset \mathrm{Im}\, f^2 \Rightarrow \underbrace{\{f(x) \mid x \in E\}}_{\mathrm{Im}\, f} \subset \{f(f(x)) \mid x \in E\}$

$\{f(y) \mid y \in \mathrm{Im}\, f\}$

Exo 5 sol de prof

$1 \Rightarrow 3$

Soit $x \in \operatorname{Ker} f \cap \operatorname{Im} f$

$f(x) = 0$ et $\exists\, y \in E,\ f(y) = x$

$f^2(y) = f(f(y)) = f(x) = \vec{0} \Rightarrow y \in \operatorname{Ker} f$

Or $\operatorname{Ker}(f^2) = \operatorname{Ker} f$, $y \in \operatorname{Ker} f \Rightarrow x = f(y) = 0$

$\dim(\operatorname{Ker} f \oplus \operatorname{Im} f) = \dim \operatorname{Ker} f + \dim \operatorname{Im} f = \dim E$

$\operatorname{Ker} f \oplus \operatorname{Im} f = E$

$3 \Rightarrow 2$

$\operatorname{Im} f^2 \subseteq \operatorname{Im} f$

$y \in \operatorname{Im} f \Rightarrow \exists\, u,\ f(u) = y$

$x \in \operatorname{Ker} f$, $v \in \operatorname{Im} f$ tq $u = x + v$ donc

$f(u) = f(x+v) = f(x) + f(v) = f(v)$

$y = f(x) = f^2(v)$, $y \in \operatorname{Im} f^2$

$2 \Rightarrow 1$ $\operatorname{Ker} f \subseteq \operatorname{Ker} f^2$

$$\begin{aligned}\dim \operatorname{Ker} f &= \dim E - \dim \operatorname{Im} f \\ &= \dim E - \dim \operatorname{Im} f^2 \\ &= \dim \operatorname{Ker} f^2\end{aligned}$$

donc $\dim \operatorname{Ker} f = \dim \operatorname{Ker} f^2$

Exercice 4

1 => 2) $a \neq b \in \mathbb{R}$

On suppose $(f - aI)\circ(f - bI) = 0$

montrons que $\mathrm{Ker}(f-aI) \oplus \mathrm{Ker}(f-bI) = E$

$x \in \mathrm{Ker}(f-aI) \cap \mathrm{Ker}(f-bI)$

$\begin{cases} f(x) = ax \\ f(x) = bx \end{cases}$ $\Rightarrow ax = bx \Rightarrow x = 0$ car $a \neq b$

$x \in E$

$u := f(x) - bx \in \mathrm{Ker}(f-aI)$

$(f-aI)(u) = (f-aI)\circ(f-bI)(x) = \vec{0}$

$v := f(x) - ax \in \mathrm{Ker}(f-bI)$

$(f-bI)(\vec{v}) = \vec{0}$

$u - v = (-a+b)x$

$x = \underbrace{\frac{u}{b-a}}_{\in \mathrm{Ker}(f-aI)} + \frac{v}{a-b} \in \mathrm{Ker}(f-bI)$

$\Rightarrow \mathrm{Ker}(f-aI) \oplus \mathrm{Ker}(f-bI) = E$

3 => 2

On suppose que $\mathrm{Ker}(f-aI) + \mathrm{Ker}(f-bI) = E$

Soit α base de $\mathrm{Ker}(f-aI)$
β base de $\mathrm{Ker}(f-bI)$

$\eta = \alpha \cup \beta$

$x \in \operatorname{Ker}(f - aI)$

$f(x) = ax$

$x \in \operatorname{Ker}(f - bI) \qquad f(x) = bx$

$$f = \begin{array}{cc} \begin{array}{cc} \alpha & \beta \end{array} & \\ \left(\begin{array}{c|c} \begin{matrix} a & & 0 \\ & \ddots & \\ 0 & & a \end{matrix} & 0 \\ \hline 0 & \begin{matrix} b & & 0 \\ & \ddots & \\ 0 & & b \end{matrix} \end{array}\right) & \begin{array}{c} \alpha \\ \beta \end{array} \end{array}$$

2 => 1]

$$f - aI = \left(\begin{array}{c|c} 0 & 0 \\ \hline & \end{array}\right)$$