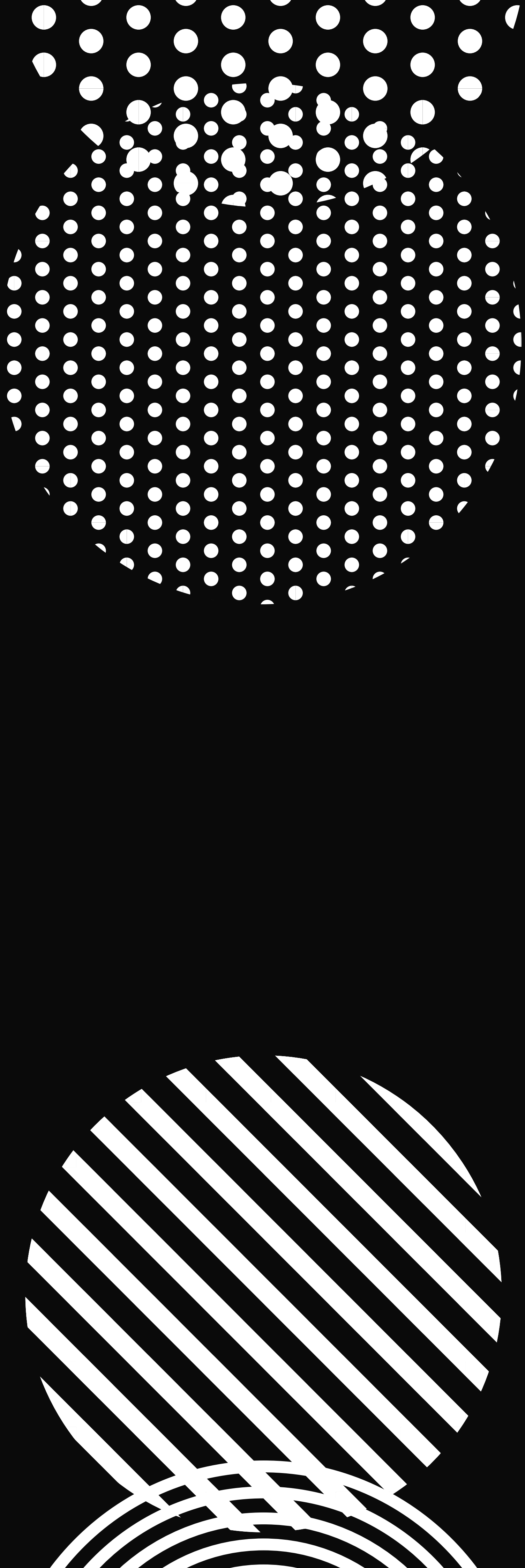

Exercice 4

1) La dimension est : 3

2) (a) $\phi(1) = 2\cdot 1 - (X-1)\cdot 0 = 2$

$\phi(X) = 2\cdot X - (X-1)\cdot 1 = 2X - X + 1 = X+1$

$\phi(X^2) = 2X^2 - (X-1)2X = 2X^2 - 2X^2 + 2X = 2X$

donc $Mat_E(\phi) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{array}{ccc} x & y & z \end{array} \left(\begin{array}{ccc|c} 2 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x = z \\ y = -2z \end{array}$

Donc $Ker(\phi) = Vect\left(\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}\right)$

(c) $Im\,\phi = Vect\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right)$

(d) $\dim Im\,\phi = 2$
$\dim Ker\,\phi = 1$

Soit $x \in Im\,\phi \cap Ker\,\phi$

Donc $x = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = c\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \Rightarrow a = c \quad b = -2c \quad c = 0$
donc $x = 0$

Alors $Ker\,\phi \cap Im\,\phi = \{0\}$

par conséquent, $E = Im\,\phi \cap Ker\,\phi$

3)

(a) Soient $P, Q, R = z_0 + z_1X + z_2X^2$

$\langle P+\lambda R, Q\rangle = (a_0+\lambda z_0)b_0 + (a_1+\lambda z_1)b_1 + (a_2+\lambda z_2)b_2$

$= a_0b_0 + \lambda z_0b_0 + a_1b_1 + \lambda z_1b_1 + a_2b_2 + \lambda z_2b_2$

$= (a_0b_0 + a_1b_1 + a_2b_2) + \lambda(z_0b_0 + z_1b_1 + z_2b_2)$
$= \langle P, Q\rangle + \lambda\langle R, Q\rangle$

par la commutativité le l'application $\langle P, Q\rangle = \langle Q, P\rangle$
donc $\langle P, Q+\lambda R\rangle = \langle P, Q\rangle + \lambda\langle P, R\rangle$
Donc $\langle\cdot,\cdot\rangle$ est bilinéaire et symmétrique.

$\langle P, P\rangle = a_0^2 + a_1^2 + a_2^2$ donc $\forall P \in E, \ \langle P, P\rangle \geq 0$

si $\langle P, P\rangle = 0 \Rightarrow a_0^2 + a_1^2 + a_2^2 = 0 \Rightarrow a_0 = a_1 = a_2 = 0$

Donc $\langle \cdot, \cdot \rangle$ est bien un produit scalaire.

b) Soit $P(x) = 0$ $\forall X$ donc $P(1) = 0$

Alors F admet l'élément nul.

$\forall P \in F$ $P(X) - P(X) = 0$ donc tout élément admet une inverse.

$\forall P, Q \in F$ et $\lambda \in \mathbb{R}$ $(P + \lambda Q)X = P(X) + \lambda Q(X)$

On a $P(1) + \lambda Q(1) = 0 + \lambda \cdot 0 = 0$ donc $P + \lambda Q \in F$
donc F est stable par addition et multiplication, donc F est un sous-espace vectoriel.

$a_0 + a_1 + a_2 = 0 \Rightarrow a_0 = -a_1 - a_2$

donc la solution est: $P = -a_1 - a_2 + a_1 X + a_2 X^2$ et la base de F est: $\begin{pmatrix} -a_1 - a_2 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(c) $e_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$$v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -2 + 1 \\ 0 - 1 \\ 2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$$

$$\|v_2\| = \frac{1}{2}\sqrt{6} = \frac{\sqrt{2}\sqrt{3}}{\sqrt{2}\sqrt{2}} = \sqrt{\frac{3}{2}}$$

Donc $e_2 = \frac{\sqrt{2}}{\sqrt{3}} \frac{1}{2} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

d) $\left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \right\rangle \frac{1}{6} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}$

$$\frac{1}{2} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -3 + 1 \\ 3 + 1 \\ 0 - 2 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

$$= \frac{1}{3}(-1 + 2X - X^2)$$

Exercice 3

$F = \{(x,y,t,z) : \mathbb{R}^4 \mid x - y - 2t = 0 \text{ et } y - z - 2t = 0\}$

1)
$$\begin{cases} x - y - 2t = 0 \\ y - z - 2t = 0 \end{cases} \quad \begin{pmatrix} 1 & -1 & -2 & 0 \\ 0 & 1 & -2 & -1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & -4 & -1 \\ 0 & 1 & -2 & -1 \end{pmatrix}$$

$$x = y + 2t \qquad \begin{pmatrix} y+2t \\ y \\ t \\ y-2t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$
$$z = y - 2t$$

2)
$$w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\|w_1\| = \sqrt{2}$$

Donc $e_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$

$$w_2 = \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \end{pmatrix} - \left\langle \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4 \\ 2 \\ 1 \\ 0 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$
$$= \begin{pmatrix} 4-2 \\ 2-2 \\ 1 \\ 0-2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$$e_2 = \frac{1}{3} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

donc $F = \mathrm{Vect}\left(\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \frac{1}{3} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right)$

3)
$$p_F(v) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\rangle \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} \right\rangle \frac{1}{9} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} + \frac{1}{9} \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 9+2 \\ 9 \\ 0+1 \\ 9-2 \end{pmatrix} = \frac{1}{9} \begin{pmatrix} 11 \\ 9 \\ 1 \\ 7 \end{pmatrix}$$

$$d(u,F) = \|v - p_F(v)\| = \frac{1}{9} \left\| \begin{pmatrix} 9 \\ 9 \\ 9 \\ 9 \end{pmatrix} - \begin{pmatrix} 11 \\ 9 \\ 1 \\ 7 \end{pmatrix} \right\| = \frac{1}{9} \left\| \begin{pmatrix} -2 \\ 0 \\ 8 \\ 2 \end{pmatrix} \right\|$$
$$= \frac{1}{9} \sqrt{4 + 64 + 4}$$
$$= \sqrt{72} \quad \frac{1}{9} = \frac{2\sqrt{2}}{3}$$

Exercice 2

Soit $k \in \operatorname{Ker} f$ $\quad \forall\, y \in \operatorname{Im} f$ i.e $\exists\, x \in E$ t.q $f(x) = y$

$$\underset{0}{\langle \underset{\shortparallel}{f(k)}, x \rangle} = 0 = \langle k, \underset{y}{f(x)} \rangle \quad \text{i.e} \quad k \perp y$$

Donc $\operatorname{Im} f \subset (\operatorname{Ker} f)^{\perp}$

Montrons que $(\operatorname{Ker} f)^{\perp} \subset \operatorname{Im} f$

Soit $k' \in (\operatorname{Ker} f)^{\perp}$ donc $\forall\, k \in \operatorname{Ker} f$ $\quad \langle k', k \rangle = 0$

$$\langle f(k'), k \rangle = \langle k', f(k) \rangle = 0$$

$$\dim E = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$$
$$= \dim (\operatorname{Ker} f)^{\perp} + \dim \operatorname{Ker} f \quad \Rightarrow \dim \operatorname{Im} f = \dim (\operatorname{Ker} f)^{\perp}$$

donc $\quad \operatorname{Im} f = (\operatorname{Ker} f)^{\perp}$