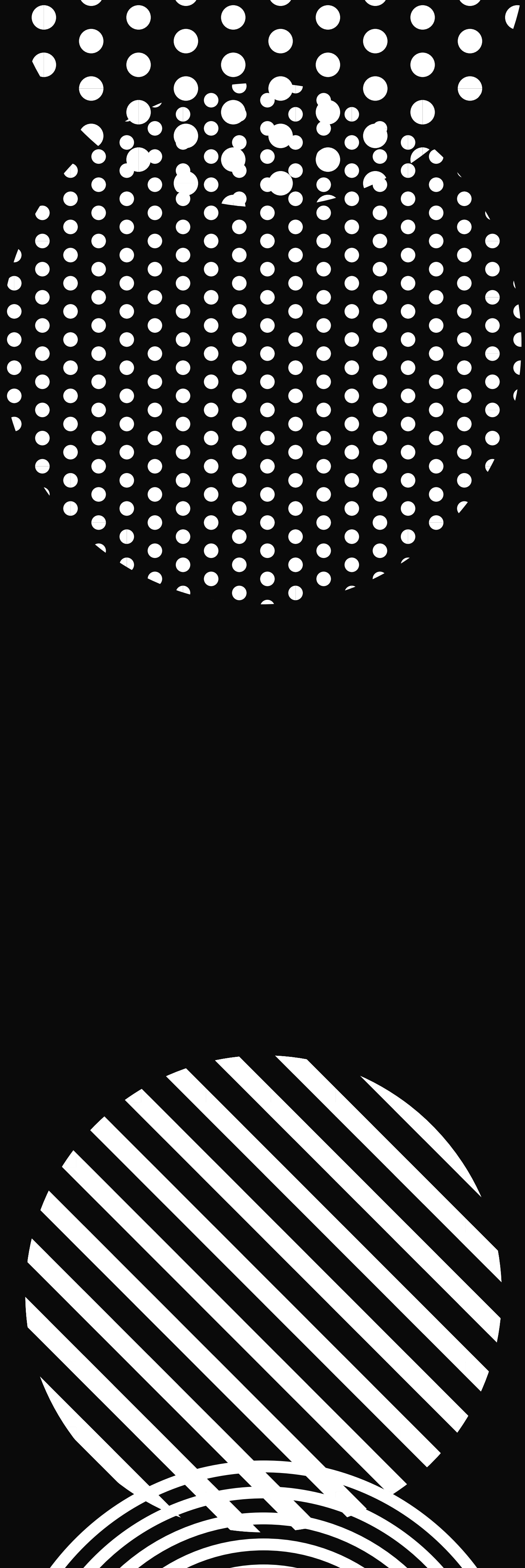

Exercice 1

$E = \text{Vect}(B)$ où $B = (e_1, e_2, e_3)$ orthonormée

$P = \{(x,y,z) \in \mathbb{R}^3 : x+y+z = 0\} = \begin{pmatrix} x \\ 0 \\ -x \end{pmatrix} + \begin{pmatrix} 0 \\ y \\ -y \end{pmatrix}$

$= \text{Vect}\Big(\underbrace{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}_{u_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}_{u_2}\Big)$

Notons B base canonique, donc $B = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right)$

(u_1, u_2) n'est pas orthonormée, on va procéder par Gram - Schmidt

$u_1' = u_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ $\quad \|u_1'\| = \sqrt{1+1} = \sqrt{2}$

$w_1 = \frac{u_1'}{\|u_1'\|} = \frac{1}{\sqrt{2}} u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$u_2' = u_2 - \langle u_2, w_1 \rangle w_1$

$= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} - \left\langle \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$\|u_2'\| = \frac{1}{2} \sqrt{1+4+1} = \frac{1}{2} \sqrt{6} = \frac{\sqrt{6}}{2}$

Donc $w_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

Donc $P = \text{Vect}\left(\frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right)$

Soit $X \in E$ $\quad p_P(X) = \langle X, w_1 \rangle w_1 + \langle X, w_2 \rangle w_2$ car (w_1, w_2) orthonormée

$p_P(e_1) = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\rangle \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$= \frac{1}{2} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{6} \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$

$= \frac{1}{6} \begin{pmatrix} 3 + 1 \\ 0 - 2 \\ -3 + 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 4 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$

$$p_p(e_2) = p_p\begin{pmatrix}0\\1\\0\end{pmatrix} = \underbrace{\left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}}}_{0} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \left\langle \begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\rangle \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$= \frac{2}{6}\begin{pmatrix}-1\\2\\-1\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\ \frac{2}{3}\\ -\frac{1}{3}\end{pmatrix}$$

$$p_p(e_3) = p_p\begin{pmatrix}0\\0\\1\end{pmatrix} = \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}\right\rangle \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix} + \left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\2\\-1\end{pmatrix}\right\rangle \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix}1\\0\\-1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}-3+1\\0-2\\3+1\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-2\\-2\\4\end{pmatrix} = \begin{pmatrix}-\frac{1}{3}\\ -\frac{1}{3}\\ \frac{2}{3}\end{pmatrix}$$

Donc $\mathrm{Mat}_B(p) = \begin{pmatrix}\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3}\\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3}\\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3}\end{pmatrix} = \frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}$

<u>Exercice 1</u> version 2

$P = \{(x,y,z) \in \mathbb{R}^3 : x+y+z=0\}$

$= \{(x,y,z) \text{ tq } \langle \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}x\\y\\z\end{pmatrix}\rangle = 0\}$

$=$ " tous les vecteurs orthogonaux à $\mathrm{Vect}\begin{pmatrix}1\\1\\1\end{pmatrix}$ "

$P = \mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)^\perp$

Donc $E = \underbrace{\mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)}_{P^\perp} \overset{\perp}{\oplus} \underbrace{\mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)^\perp}_{P}$ car E de dim. finie.

Donc $\forall X \in E$, $X = p_P(X) + p_{P^\perp}(X)$

$$p_{P^\perp}(X) = \frac{\langle X, \begin{pmatrix}1\\1\\1\end{pmatrix}\rangle}{\left\|\begin{pmatrix}1\\1\\1\end{pmatrix}\right\|^2}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\langle X, \begin{pmatrix}1\\1\\1\end{pmatrix}\rangle \begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}^T X$$

Donc $p_P(X) = X - p_{P^\perp}(X) = \left(\mathrm{Id} - \frac{1}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}(1\ 1\ 1)\right) X$

$$= \left(\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} - \frac{1}{3}\begin{pmatrix}1 & 1 & 1\\ 1 & 1 & 1\\ 1 & 1 & 1\end{pmatrix}\right) X$$

$$= \underbrace{\frac{1}{3}\begin{pmatrix}2 & -1 & -1\\ -1 & 2 & -1\\ -1 & -1 & 2\end{pmatrix}}_{P} X$$

Exercice 2

Soient E esp. euclidien. A, B s.e.v de E.

Supposons $A \subset B$.

Soit $a \in A$, alors $a \in B$

$\forall b' \in B^{\perp}$ on a $\langle a, b' \rangle = 0$ car $a \in B$

donc $b' \in A^{\perp}$. Alors $B^{\perp} \subset A^{\perp}$

Exercice 3

$A^H A = I_n$

$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}$$

$\Rightarrow a^2 = 1$

$b^2 + d^2 = 1 \Rightarrow d^2 = 1$

$ab = 0 \Rightarrow b = 0$

$ac = 0 \Rightarrow c = 0$

$de = 0 \Rightarrow e = 0$

$f^2 = 1 \Rightarrow$

$$\begin{pmatrix} a & 0 & 0 \\ b & d & 0 \\ c & e & f \end{pmatrix}$$

A avec les éléments diagonaux égaux à ± 1 et 0 ailleurs.

Exercice 4

$\mathbb{R}_2[X] = \mathrm{Vect}(1, X, X^2)$

$\forall P, Q \in \mathbb{R}_2[X]$ $\quad \langle P, Q \rangle = \int_{-1}^{1} P(T)Q(T)\,dt$

$\langle 1, 1 \rangle = \int_{-1}^{1} 1\,dt = [t]_{-1}^{1} = 1 + 1 = 2$

$\|1\| = \sqrt{\langle 1, 1 \rangle} = \sqrt{2}$ donc $w_1 = \frac{1}{\sqrt{2}} 1 = \frac{1}{\sqrt{2}}$

$$u_2' = u_2 - \langle u_2, w_1 \rangle w_1$$

$$= X - \langle X, \tfrac{1}{\sqrt{2}} \rangle \tfrac{1}{\sqrt{2}}$$

$$= X - \tfrac{1}{\sqrt{2}} \int_{-1}^{1} t \, dt \ \tfrac{1}{\sqrt{2}} = X$$

$$\|X\| = \left(\int_{-1}^{1} t^2 dt\right)^{\frac{1}{2}} = \left(\left[\tfrac{t^3}{3}\right]_{-1}^{1}\right)^{\frac{1}{2}} = \frac{\sqrt{2}}{\sqrt{3}}$$

$$w_2 = \sqrt{\tfrac{3}{2}}\, X$$

$$u_3 = X^2 - \langle X^2, w_1 \rangle w_1 - \langle X^2, w_2 \rangle w_2$$

$$\langle X^2, \tfrac{1}{\sqrt{2}} \rangle = \tfrac{1}{\sqrt{2}} \int_{-1}^{1} t^2 dt = \tfrac{1}{\sqrt{2}} \, \tfrac{2}{3} = \tfrac{\sqrt{2}}{3}$$

$$\langle X^2, w_1 \rangle w_1 = \tfrac{\sqrt{2}}{3} \, \tfrac{1}{\sqrt{2}} = \tfrac{1}{3}$$

$$\langle X^2, w_2 \rangle = \langle X^2, \sqrt{\tfrac{3}{2}}\, X \rangle = \sqrt{\tfrac{3}{2}} \int_{-1}^{1} t^3 dt = 0 \quad \text{car impair}$$

Donc $u_3 = X^2 - \frac{1}{3}$

$$w_3 = \frac{1}{\|u_3\|} u_3$$

$$\|u_3\| = \sqrt{\langle u_3, u_3 \rangle} = \left(\int_{-1}^{1} \left(t^2 - \tfrac{1}{3}\right)\left(t^2 - \tfrac{1}{3}\right) dt\right)^{\frac{1}{2}}$$

$$= \left(\int_{-1}^{1} t^4 - \tfrac{1}{3}t^2 - \tfrac{1}{3}t^2 + \tfrac{1}{9} dt\right)^{\frac{1}{2}}$$

$$= \left(\left[\tfrac{t^5}{5} - \tfrac{2t^3}{9} + \tfrac{t}{9}\right]_{-1}^{1}\right)^{\frac{1}{2}} = \left(\tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9} + \tfrac{1}{5} - \tfrac{2}{9} + \tfrac{1}{9}\right)^{\frac{1}{2}}$$

$$= \left(\tfrac{2}{5} - \tfrac{4}{9} + \tfrac{2}{9} = \tfrac{2}{5} - \tfrac{2}{9}\right)^{\frac{1}{2}}$$

$$= \left(\frac{18 - 10}{45}\right)^{\frac{1}{2}} = \left(\frac{8}{45}\right)^{\frac{1}{2}} = \frac{2\sqrt{2}}{3\sqrt{5}}$$

Donc $w_3 = \frac{3\sqrt{5}}{2\sqrt{2}}\left(X^2 - \frac{1}{3}\right)$

Donc la base orthonormé de $\mathbb{R}_2[X]$ est:

$$\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}X, \frac{3\sqrt{5}}{2\sqrt{2}}\left(X^2 - \frac{1}{3}\right)\right)$$

$$\begin{array}{c|ccc} & 1 & X & X^2 \\ 1 & \multicolumn{3}{l}{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \\ X & & & \\ X^2 & & & \end{array}$$

Base orthonormé

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{\sqrt{5}}{2\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & \\ 0 & 0 & \frac{3\sqrt{5}}{2\sqrt{2}} \end{pmatrix}$$

Exercice 5

Soit $v = (0,0,1)$ $\quad F = Vect((2,1,1),(3,2,1))$

$\mathbb{R}^3 = F \oplus F^\perp$ donc $v = p_F(v) + p_{F^\perp}(v)$ et $\|p_{F^\perp}(v)\|$ est $d(v,F)$

Pour cela on va ortho normaliser par Gram-Schmidt $\left(\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right)$

$u_1 = \begin{pmatrix}2\\1\\1\end{pmatrix}$ $\quad \|u_1\| = \sqrt{4+1+1} = \sqrt{6}$

donc $w_1 = \frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}$

$$u_2 = \begin{pmatrix}3\\2\\1\end{pmatrix} - \left\langle \begin{pmatrix}3\\2\\1\end{pmatrix}, w_1 \right\rangle w_1 = \begin{pmatrix}3\\2\\1\end{pmatrix} - \left\langle \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\rangle \frac{1}{\sqrt{6}} \frac{1}{\sqrt{6}} \begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}3\\2\\1\end{pmatrix} - (6+2+1)\frac{1}{6}\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} - \frac{9}{6}\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix}18 & - & 18\\12 & & 9\\6 & & 9\end{pmatrix} = \frac{1}{6}\begin{pmatrix}0\\3\\-3\end{pmatrix}$$

$$\|u_2\| = \sqrt{9+9}\ \frac{1}{6} = 3\sqrt{2}\ \frac{1}{6} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Donc $w_2 = \frac{u_2}{\|u_2\|} = \frac{\sqrt{2}}{6}\begin{pmatrix}0\\3\\-3\end{pmatrix}$

Donc $F = Vect\left(\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}, \frac{\sqrt{2}}{6}\begin{pmatrix}0\\3\\-3\end{pmatrix}\right)$

$p_F(v) = \langle v, w_1\rangle w_1 + \langle v, w_2\rangle w_2$

$\langle v, w_1\rangle = \frac{1}{\sqrt{6}}\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}2\\1\\1\end{pmatrix}\right\rangle = \frac{1}{\sqrt{6}}$

$\langle v, w_2\rangle = \frac{\sqrt{2}}{6}\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}0\\3\\-3\end{pmatrix}\right\rangle = \frac{-3\sqrt{2}}{6} = -\frac{\sqrt{2}}{2} = -\frac{1}{\sqrt{2}}$

$p_F(v) = \frac{1}{6}\begin{pmatrix}2\\1\\1\end{pmatrix} - \frac{1}{6}\begin{pmatrix}0\\3\\-3\end{pmatrix} = \frac{1}{6}\begin{pmatrix}2\\-2\\4\end{pmatrix}$

$v - p_F(v) = \frac{1}{6}\begin{pmatrix}0\\0\\6\end{pmatrix} - \frac{1}{6}\begin{pmatrix}2\\-2\\4\end{pmatrix} = \frac{1}{6}\begin{pmatrix}-2\\2\\2\end{pmatrix}$

$\|v - p_F(v)\| = \frac{1}{6}\sqrt{4+4+4} = \frac{\sqrt{12}}{6} = \frac{2\sqrt{3}}{6} = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} = d(v, F)$

Exercice 6

Or $\forall 1 \leq i \leq n$, $\|e_i\| = 1$, il suffit de montrer que les vecteurs sont orthogonaux et qu'ils engendrent E.

Pour $1 \leq i, j \leq n$ $\quad \|e_i + e_j\|^2 = \sum_{k=1}^{n} \langle e_k, e_i + e_j\rangle^2$

$$= \sum_{k=1}^{n} \left(\langle e_k, e_i\rangle + \langle e_k, e_j\rangle\right)^2$$

$\|e_i + e_j\|^2 = \|e_i\|^2 + \|e_j\|^2 + 2\langle e_i, e_j\rangle = \sum_{k=1}^{n} \langle e_k, e_i\rangle^2 + 2\langle e_k, e_i\rangle\langle e_k, e_j\rangle + \langle e_k, e_j\rangle^2$

$$= \|e_i\|^2 + \|e_j\|^2 + 2\sum_{k=1}^{n} \langle e_k, e_i\rangle\langle e_k, e_j\rangle$$

$\Rightarrow \quad \langle e_i, e_j\rangle = \sum_{k=1}^{n} \langle e_k, e_i\rangle\langle e_k, e_j\rangle \quad \Rightarrow \quad \langle e_i, e_j\rangle = 0$

Donc $(e_1, \dots, e_n)$ sont deux à deux orthogonaux.

Montrons que $\underbrace{(e_1, \dots, e_n)}_{\mathcal{B}}$ engendrent E.

Supposons par l'absurde que $\mathcal{B}$ n'engendre pas E, alors $\exists\ F \subset E$ s.e.v tq $F = \text{vect}(\mathcal{B})$

Donc $\exists\ F^\perp$ s.e.v de E tq $E = F \overset{\perp}{\oplus} F^\perp$

$\exists x \in F^\perp$ donc $\langle x, y \rangle = 0 \quad \forall y \in F$

Or $\forall i \quad e_i \in F$ donc $\langle x, e_i \rangle = 0$

$\Rightarrow \|x\|^2 = 0$

mais la norme 0 force à $x = \vec{0}$

donc $F^\perp = \{0\} \Rightarrow \underline{F = E}$

Donc $(e_1, \dots, e_n)$ est bien une base orthonormale.

Exercice 7

1. Soit A une matrice de E dans B.

Donc $\forall j \quad f(e_j) = \sum_{i=1}^{n} a_{i,j} e_i$

$\forall\ x, y \in E \qquad \langle x, y \rangle = x^T y$

$\langle f(x), y \rangle = \langle x, f(y) \rangle \Rightarrow (Ax)^T y = x^T A y$

$\Rightarrow x^T A^T y = x^T A y$

$\Rightarrow A^T = A$

Aussi $\langle f(e_i), e_j \rangle = \underbrace{\langle e_i, f(e_j) \rangle}$

$= \sum_{k=1}^{n} a_{k,i} \langle e_i, e_j \rangle \qquad \sum_{k=1}^{n} a_{k,j} \langle e_i, e_j \rangle$

$= a_{j,i} \qquad = a_{i,j}$

Ce qui conclut que A est symétrique

Soit $x \in \text{Ker}(f)$, $y \in \text{Im}(f)$ donc $\exists y'$ tq $f(y') = y$

$\underbrace{\langle f(x), y' \rangle}_{=0} = \langle x, f(y') \rangle = \langle x, y \rangle$ donc x et y sont orthogonaux.

$\forall\ z \in \text{Ker}(f) \cap \text{Im}(f)$

$f(z) = 0$, donc $\text{Ker}(f) \cap \text{Im}(f) = \{\vec{0}\}$

donc ils sont bien ~~orthogonaux~~.

Donc $\underline{\underline{\text{Ker}(f)^\perp \oplus \text{Ker}(f) = E}}$

$\Rightarrow \underbrace{\dim(\text{Ker}(f)^\perp)} = \dim(E) - \dim(\text{Ker} f) = \dim(\text{Im}(f))$

$\dim(\text{Im}(f)) + \dim(\text{Ker}(f)) = \dim(E)$

$\Rightarrow \text{Im}(f) \oplus \text{Ker} f = E$

Exercice 8

$\{\underset{u_1}{1}, \underset{u_2}{x}\}$ est une base de $F = Vect(1,x)$

Construison une BON de F

$\tilde{u}_1 = 1 \qquad \|\tilde{u}_1\| = \left(\int_0^1 1\cdot 1\, dt\right)^{\frac{1}{2}} = \left([t]_0^1\right)^{\frac{1}{2}} = \sqrt{1} = 1$

Donc $w_1 = \frac{\tilde{u}_1}{\|\tilde{u}_1\|} = 1$

$\tilde{u}_2 = u_2 - \langle u_2, w_1\rangle w_1$

$= x - \langle x, 1\rangle 1$

$\langle x, 1\rangle = \int_0^1 t\, dt = [\frac{t^2}{2}]_0^1 = \frac{1}{2}$

$\tilde{u}_2 = x - \frac{1}{2}$

$\|\tilde{u}_2\| = \left(\int_0^1 (t-\frac{1}{2})(t-\frac{1}{2})\, dt\right)^{\frac{1}{2}} = \left(\int_0^1 t^2 - \frac{1}{2}t - \frac{1}{2}t + \frac{1}{4} dt\right)^{\frac{1}{2}}$

$= \left(\int_0^1 t^2 - t + \frac{1}{4}\, dt\right)^{\frac{1}{2}}$

$= \left([\frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{4}t]_0^1\right)^{\frac{1}{2}}$

$= \left(\frac{1}{3} - \frac{1}{2} + \frac{1}{4}\right)^{\frac{1}{2}} = \left(\frac{4}{12} - \frac{6}{12} + \frac{3}{12}\right)^{\frac{1}{2}} = \sqrt{\frac{1}{12}}$

$= \frac{1}{2\sqrt{3}}$

$w_2 = 2\sqrt{3}(x - \frac{1}{2})$

$F = Vect(1, 2\sqrt{3}(x - \frac{1}{2}))$

Alors $p_F(x^2) = \langle x^2, 1\rangle 1 + \langle x^2, 2\sqrt{3}(x-\frac{1}{2})\rangle 2\sqrt{3}(x - \frac{1}{2})$

$\langle x^2, 1\rangle = \int_0^1 t^2 dt = [\frac{t^3}{3}]_0^1 = \frac{1}{3}$

$\langle x^2, 2\sqrt{3}(x-\frac{1}{2})\rangle = 2\sqrt{3}\langle x^2, x - \frac{1}{2}\rangle = 2\sqrt{3}\int_0^1 t^2(t-\frac{1}{2})dt$

$= 2\sqrt{3}\int_0^1 t^3 - \frac{t^2}{2}\, dt$

$= 2\sqrt{3}[\frac{t^4}{4} - \frac{t^3}{6}]_0^1$

$= 2\sqrt{3}(\frac{1}{4} - \frac{1}{6})$

$= 2\sqrt{3}(\frac{3}{12} - \frac{2}{12}) = \frac{2\sqrt{3}}{12} = \frac{\sqrt{3}}{6}$

$$p_F(x^2) = \frac{1}{3} + \frac{\sqrt{3}}{6}\left(2\sqrt{3}\left(x - \frac{1}{2}\right)\right)$$

$$= \frac{1}{3} - \frac{2\cdot 3}{6}\left(x - \frac{1}{2}\right)$$

$$= \frac{1}{3} + x - \frac{1}{2} = \frac{2}{6} - \frac{3}{6} + x = x - \frac{1}{6}$$

Exercice 5 version 2

$\begin{pmatrix}-1\\1\\1\end{pmatrix}$ est ortogonale à $\left(\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right)$

Alors $\underbrace{\mathrm{Vect}\begin{pmatrix}-1\\1\\1\end{pmatrix}}_{=:P^\perp} \perp \mathrm{Vect}\left(\begin{pmatrix}2\\1\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}\right) =: P$

Donc $\forall v \in E \qquad v = p_P(v) + p_{P^\perp}(v)$

$$p_{P^\perp}(v) = \frac{\left\langle v, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rangle}{\left\|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\|^2}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\left\langle \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix}\right\rangle = 1$$

$$\left\|\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\|^2 = \sqrt{1+1+1}^{\,2} = \sqrt{3}^{\,2} = 3$$

$$p_{P^\perp}(v) = \frac{1}{3}\begin{pmatrix}-1\\1\\1\end{pmatrix}$$

$$\|p_{P^\perp}(v)\| = \frac{1}{3}\sqrt{1+1+1} = \frac{\sqrt{3}}{3}$$

$$\begin{pmatrix} a & 0 \\ b & d \end{pmatrix}\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} a^2 & ab \\ ab & b^2+d^2 \end{pmatrix}$$

t

$\downarrow$

$$\begin{pmatrix} t_{1,1} & 0 & \cdots & 0 \\ t_{1,2} & t_{2,2} & \cdots & 0 \\ \vdots & 0 & \ddots & \vdots \\ \vdots & \vdots & & \vdots \\ t_{1,n} & 0 & \cdots & t_{n,n} \end{pmatrix}\begin{pmatrix} t_{1,1} & t_{1,2} & \cdots & t_{1,n} \\ 0 & t_{2,2} & \cdots & t_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & t_{n,n} \end{pmatrix}$$

i : ligne

j : colonne

$$\sum_{k=1}^{n} t_{k,i}\, t_{k,j} = 0 \quad \text{si } i \neq j$$