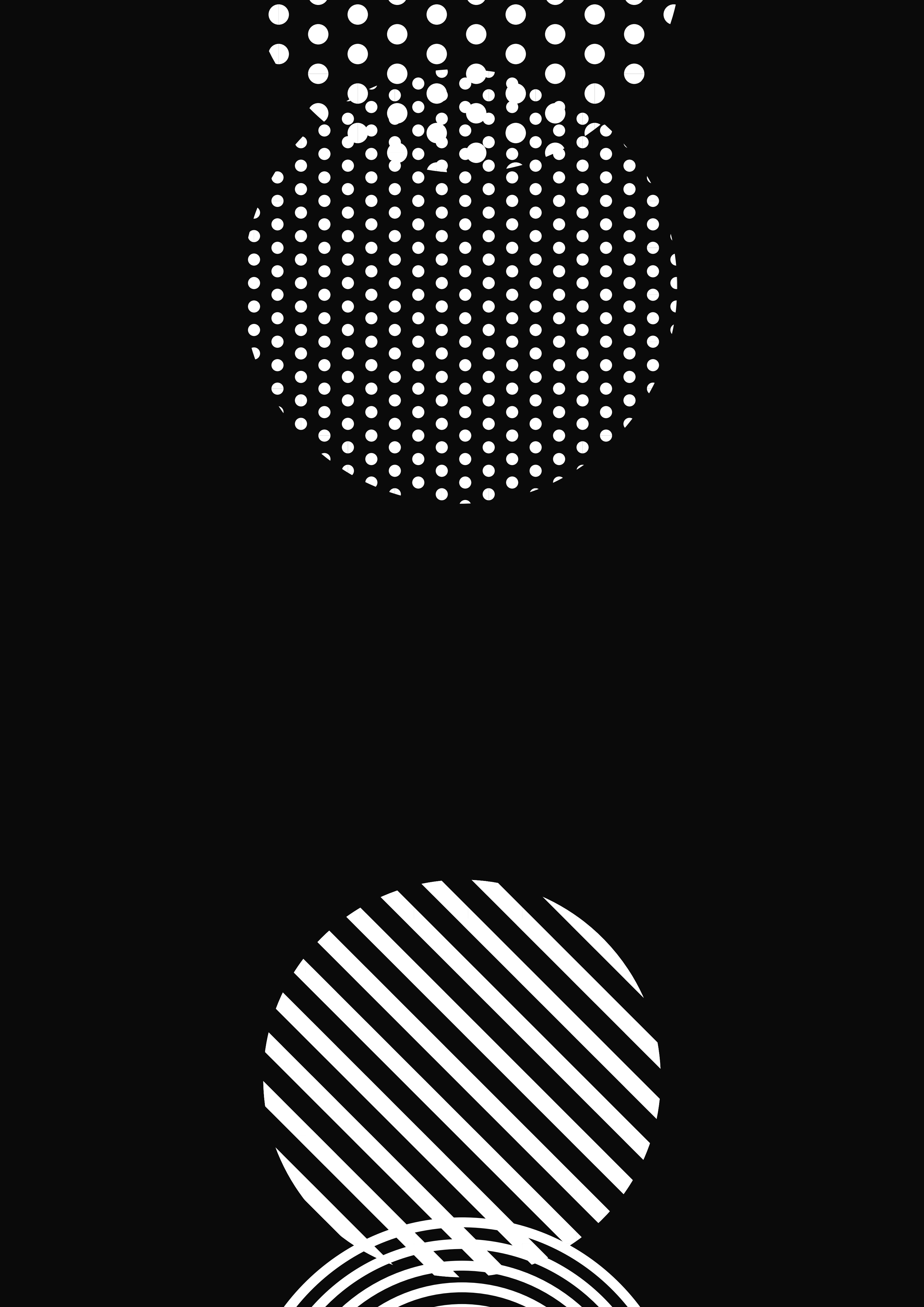

Exercice 1

$X = (x_1, \dots, x_d) \in \mathbb{R}^d$

$\|X\|_2 = \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$

$\|X\|_1 = \sum_{i=1}^{d} |x_i|$

$\|X\|_\infty = \max_{1 \le i \le d} |x_i|$

1) $\|X\|_1 = \sum_{i=1}^{d} |x_i| \ge 0$ car $\forall i \quad |x_i| \ge 0$

$\|\lambda X\|_1 = \sum_{i=1}^{d} |\lambda x_i| = \sum_{i=1}^{d} |\lambda| |x_i| = |\lambda| \sum_{i=1}^{d} |x_i| = |\lambda| \|X\|_1$

$\|X + Y\| = \sum_{i=1}^{d} |x_i + y_i| \qquad |x_i + y_i| \le |x_i| + |y_i|$

$\Rightarrow \sum_{i=1}^{d} |x_i + y_i| \le \sum_{i=1}^{d} |x_i| + |y_i| = \sum_{i=1}^{d} |x_i| + \sum_{i=1}^{d} |y_i|$

$= \|X\|_1 + \|Y\|_1$

$\|X\|_1 = 0 \Rightarrow \sum_{i=1}^{d} |x_i| = 0 \Rightarrow |x_1| + \dots + |x_d| = 0 \Rightarrow |x_1| = |x_2| = \dots = |x_d| = 0$

car valeur absolue ne peut pas être négative.

2) $\|X\|_\infty = \max_{1 \le i \le d} |x_i|$

$\|\lambda X\|_\infty = \max_{1 \le i \le d} |\lambda x_i| = \max_{1 \le i \le d} |\lambda| |x_i| = |\lambda| \max_{1 \le i \le d} |x_i|$

$= |\lambda| \|X\|_\infty$

$\|X + Y\|_\infty = \max_{1 \le i \le d} |x_i + y_i| \le \max_{1 \le i \le d} (|x_i| + |y_i|) = \max_{1 \le i \le d} |x_i| + \max_{1 \le i \le d} |y_i|$

$= \|X\|_\infty + \|Y\|_\infty$

$\|X\|_\infty \geq 0$ car $|x_i| \geq 0 \quad \forall i$

$\|X\|_\infty = 0 \Rightarrow$ 0 est la plus grande valeur parmi toutes $|x_i|$, comme $|x_i|$ toujours positive, toute $|x_i| = 0$, donc $X = 0$

2) $\|X\|_\infty = \max_{1 \leq i \leq d} |x_i|$

$\|X\|_1 = \sum_{i=1}^{d} |x_i|$ soit $|x_j|$ est max des $|x_i|$

donc $\|X_1\| = |x_j| = \underbrace{|x_1| + \dots + |x_{j-1}| + |x_{j+1}| + \dots + |x_d|}_{\geq 0} + |x_j|$

$\Rightarrow \|X_1\| \geq \max_{1 \leq i \leq d} |x_i| = |x_j| = \|X\|_\infty$

$\forall i \in \{1, \dots, d\} \quad |x_i| \leq |x_j| =: \max_{1 \leq i \leq d} |x_i|$

$\Rightarrow |x_1| + \dots + |x_d| \leq \underbrace{|x_j| + \dots + |x_j|}_{d \text{ fois}} \Rightarrow \|X\|_1 \leq d\|X\|_\infty$

3) Notons $|x_j| := \max_{1 \leq i \leq d} |x_i|$ avec $1 \leq j \leq d$

$\|X\|_\infty^2 = |x_j|^2 = x_j^2$

$\|X\|_2^2 = \underbrace{x_1^2 + \dots + x_{j-1}^2 + x_{j+1}^2 + \dots + x_d^2}_{\geq 0} + x_j^2$ donc $\|X\|_2^2 \geq |x_j|^2$

$\Rightarrow \|X_2\| \geq \|X\|_\infty$

$\forall i, \quad x_i^2 \leq x_j^2 \Rightarrow x_1^2 + x_2^2 + \dots + x_d^2 \leq \underbrace{x_j^2 + \dots + x_j^2}_{d \text{ fois}}$

$\Rightarrow \|X\|_2^2 \leq d\|X\|_\infty^2 \Rightarrow \|X\|_2 \leq \sqrt{d}\,\|X\|_\infty \leq d\|X\|_\infty$

Exercice 2

$$\|X\|_2 = (X \cdot X)^{\frac{1}{2}}$$

$$\|X\|_2 = X \cdot X$$

$$\|X+Y\|^2 = (X+Y) \cdot (X+Y) = X \cdot (X+Y) + Y(X+Y)$$
$$= X \cdot X + X \cdot Y + Y \cdot X + Y \cdot Y$$

$$\|X-Y\|^2 = (X-Y) \cdot (X-Y) = X \cdot (X-Y) - Y(X-Y)$$
$$= X \cdot X - X \cdot Y - YX + Y \cdot Y$$

$$\|X+Y\|^2 + \|X-Y\|^2 = X^2 + X \cdot Y + Y \cdot X - X \cdot Y - Y \cdot X + Y^2 + X^2 + Y^2$$

$$= 2\|X\|^2 + 2\|Y\|^2 = 2(\|X\|^2 + \|Y\|^2)$$

Exercice 3

Soient $X, Y \in \mathbb{R}^2$

Supposons $0, X, Y$ alignées : $\delta(X,Y) = d(X,Y) = d(Y,X)$ par déf
$\delta(Y,X) = d(Y,X) = \delta(X,Y)$ de la distance.

$$\delta(X,Y) = d(X,Y) \leq d(X,Z) + d(Y,Z) \leq \underbrace{\delta(X,Z)}_{\substack{\geq d(X,Z) \\ \geq d(0,X) + d(0,Z)}} + \underbrace{\delta(Y,Z)}_{\geq d(Y,Z) \geq d(0,Y) + d(0,Z)}$$

$\delta(X,Y) = d(X,Y) \geq 0$ par déf. de distance

$\delta(X,Y) = d(X,Y) = 0 \Rightarrow X = Y$ par déf de distance

Supposons O, X, Y pas alignées :

$\delta(X,Y) = d(O,X) + d(O,Y) = d(O,Y) + d(O,X)$ car l'addition est commutative.

$\delta(Y,X) = d(O,Y) + d(O,X) = \delta(X,Y)$

$\delta(X,Y) = d(O,X) + d(O,Y) \geq d(X,Y) \leq d(X,Z) + d(Y,Z)$
$\leq d(O,Y) + d(O,Z) + d(O,X) + d(O,Z)$

$\delta(X,Z) \leq d(X,Z)$
$\leq d(O,X) + d(O,Z)$

$\delta(Y,Z) \leq d(Y,Z)$
$\leq d(O,Y) + d(O,Z)$

donc $\delta(X,Y) \leq \delta(X,Z) + \delta(Y,Z)$

$\delta(X,Y) = d(O,X) + d(O,Y) \geq 0$ car $d(O,X) \geq 0$ et $d(O,Y) \geq 0$

$\delta(X,Y) = d(O,X) + d(O,Y) = 0 \Rightarrow d(O,X) = 0$ et $d(O,Y) = 0$ car d toujours positive.

$\Rightarrow O = X$, $O = Y$ par déf de distance

$\Rightarrow X = Y$

Donc δ est bien une distance.

Exercice 4

① $d_{log}(x,y) = |\log_{10}(x y^{-1})|$

1) $d_{log}(y,x) = |\log_{10}(\frac{y}{x})| = |\log_{10}((\frac{x}{y})^{-1})|$

$= |-\log_{10}(\frac{x}{y})| = |\log_{10}(x y^{-1})|$

$= d_{log}(x,y)$

2) $d_{log}(x,z) + d_{log}(y,z)$

$= |\log_{10}(x z^{-1})| + |\log(z y^{-1})|$

$= |\log_{10}(x) - \log_{10}(z)| + |\log_{10}(z) - \log_{10}(y)|$

$\geq |\log_{10}(x) - \log_{10}(y)| = |\log_{10}(x y^{-1})| = d_{log}(x,y)$

3) $d_{log}(x,y) = |\log_{10}(x y^{-1})| \geq 0$ évidemment.

$d_{log}(x,y) = |\log_{10}(x y^{-1})| = 0 \Rightarrow x y^{-1} = 1 \Rightarrow x = y$

$d_{log}(x,x) = |\log_{10}(x \cdot x^{-1})| = 0$

②

$d_{log}(10^p, 10^q) = |\log_{10}(10^{p-q})| = |p-q|$

③ Soit $y_n = 10^n$

$d_{log}(1, y_n) = |\log_{10}(10^n)| = |n|$

$|1-10^n| = |10^n - 1| \underset{n\to+\infty}{\sim} |10^n|$

$\underline{|-n-1| \leq c|10^n - 1|}$

$\Rightarrow |n| \leq C$

$\lim_{n\to+\infty} \frac{d_{log}(1,y_n)}{|x-y|} = 0 = C$, mais $d_{log}(x,y) \leq 0$ est faux, donc $\nexists C$.

4) Même que dans 3)

$$\lim_{n \to +\infty} \frac{|x-y|}{d_{\log}(x,y)} = +\infty = C$$

Donc il n'existe pas tel $\underline{C}$.