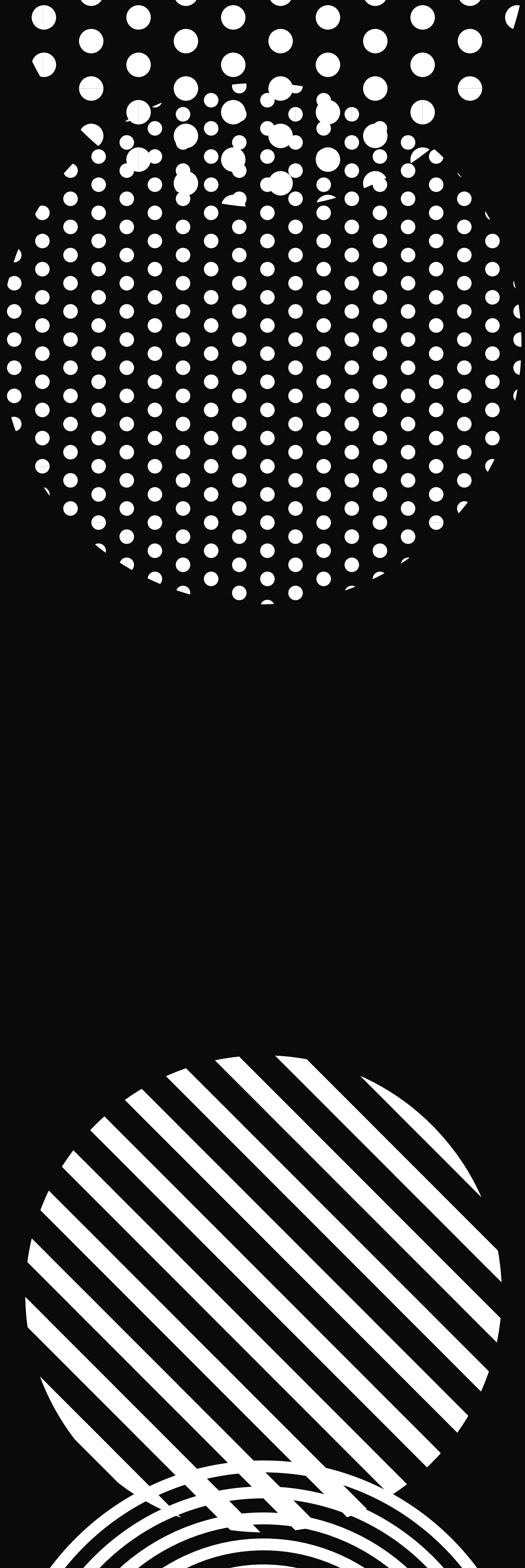

Exercice 1

$\mathbb{R}^d \qquad X = \begin{pmatrix} x^1 \\ x^2 \\ \vdots \\ x^d \end{pmatrix}$

$$|x|_2 = \sqrt{(x^1)^2 + \dots + (x^d)^2}$$

$Y = \begin{pmatrix} y^1 \\ \vdots \\ y^d \end{pmatrix} \qquad \langle X, Y \rangle := \sum_{i=1}^{d} x^i y^i$

$$|x|_2^2 = \langle x, x \rangle$$

$u_1, \dots, u_n \in \mathbb{R}^d \qquad \forall i \in [\![1, n]\!] \qquad |u_i|_2 = 1$

$\exists\ \varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ tq

$$|\varepsilon_1 u_1 + \dots + \varepsilon_n u_n|_2 \leq \sqrt{n}$$

$\varepsilon_1, \dots, \varepsilon_n$ v.a. i.i.d

$E[\varepsilon_i \varepsilon_j] = E[\varepsilon_i] E[\varepsilon_j] = 0$

$\forall i, \quad P(\varepsilon_i = -1) = P(\varepsilon_i = 1) = \frac{1}{2} \qquad E[\varepsilon_i] = 0 \qquad E[\varepsilon_i^2] = 1$

$S = \varepsilon_1 u_1 + \dots + \varepsilon_n u_n \qquad |S|_2^2 = \langle S, S \rangle = \langle \sum_{i=1}^{n} \varepsilon_i u_i, \sum_{j=1}^{n} \varepsilon_j u_j \rangle$

$$= \sum_i \sum_j \varepsilon_i \varepsilon_j \langle u_i, u_j \rangle$$

$$E[|S|_2^2] = E[\sum_i \sum_j \varepsilon_i \varepsilon_j \langle u_i, u_j \rangle] = \sum_i \sum_j E[\varepsilon_i \varepsilon_j] \langle u_i, u_j \rangle$$

$$= \sum_{i=1}^{n} \langle u_i, u_j \rangle = \sum_{i=1}^{n} |u_i|_2^2 = n$$

$$E[|S|_2^2] = \sum_{\epsilon_1, \dots, \epsilon_n \in \{-1, +1\}} \underbrace{P(\varepsilon_1 = \epsilon_1, \dots, \varepsilon_n = \epsilon_n)}_{\frac{1}{2^n}} |\epsilon_1 u_1 + \dots + \epsilon_n u_n|_2^2$$

$$= \frac{1}{2^n} \sum |\epsilon_1 u_1 + \dots + \epsilon_n u_n|_2^2 = n \implies \exists\, \epsilon_1, \dots, \epsilon_n \text{ tq } |\epsilon_1 u_1 + \dots + \epsilon_n u_n|_2^2 \leq n$$

Exercice 2

$(\Omega, \mathcal{F}, P)$ espace de proba

$$X : (\Omega, \mathcal{F}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

loi de X : mesure sur $\mathbb{R}$

$B \in \mathcal{B}(\mathbb{R})$, $\quad P_X(B) = \mathbb{P}(X^{-1}(B)) = P(X \in B)$

$$\sigma(X) = \{ X^{-1}(B), \ B \in \mathcal{B}(\mathbb{R}) \}$$

La méthode de la fonction muette

$\forall f : \mathbb{R} \to \mathbb{R}$ continue bornée, $E[f(X)] = \dots = \int_{\mathbb{R}} f(x) \boxed{\mu(dx)}$ $\int_{\mathbb{R}} f(x) \, dP_X(x)$

$\leftrightarrow$ loi de X

a) $\quad X(\omega) = a \mathbb{1}_A(\omega) + b \mathbb{1}_B(\omega)$

$$X \in \{0, a, b, a+b\}$$

Supposons que $0, a, b, a+b$ 2 à 2 distincts

$$\sigma(X) = \sigma(\{ X^{-1}(0), X^{-1}(a), X^{-1}(b), X^{-1}(a+b) \})$$
$$= \sigma(\{ (A \cup B)^c, A \setminus B, B \setminus A, A \cap B \})$$

$P(X = 0) = \mathbb{P}((A \cup B)^c)$

$P(X = a) = \mathbb{P}(A \setminus B)$

$P(X = b) = \mathbb{P}(B \setminus A)$

$P(X = a+b) = \mathbb{P}(A \cap B)$

Par ex.

si $a = b$

$$\sigma(X) = \sigma(\{ X^{-1}(0), X^{-1}(a), X^{-1}(2a) \})$$
$$= \sigma(\{ (A \cup B)^c, A \Delta B, A \cap B \})$$

$P(X = 0) = P((A \cup B)^c)$

$P(X = a) = P(A \Delta B)$

$P(X = 2a) = P(A \cap B)$

b) $\quad (\Omega, \mathcal{F}, P) = ([0,1], \mathcal{B}([0,1]), \lambda)$

$$X(\omega) = \begin{cases} 2\omega & \text{si } 0 \leq \omega \leq \frac{1}{2} \\ 1 & \text{si } \frac{1}{2} \leq \omega \leq 1 \end{cases} \qquad X : \Omega \to [0,1]$$

$$\sigma(X) = \{ X^{-1}(B), \ B \in \mathcal{B}([0,1]) \}$$

$$X^{-1}(B) = \begin{cases} \{ \omega \in [0, \frac{1}{2}[, \ 2\omega \in B \} & \text{si } 1 \notin B \\ \{ \omega \in [0, \frac{1}{2}, \ 2\omega \in B \} \cup [\frac{1}{2}, 1] & 1 \in B \end{cases}$$

- $\sigma(X) = \{ C \cup D \mid C \in \mathcal{B}([0, \frac{1}{2}]), \ D \in \{ \emptyset, [\frac{1}{2}, 1] \} \}$

• $\forall f$ continue

$$E[f(X)] = \int_\Omega f(X(\omega))\, dP(\omega)$$

$$= \int_0^{\frac{1}{2}} f(2\omega) \frac{d\lambda(\omega)}{d\omega} + \int_{\frac{1}{2}}^{1} f(1)\, d\omega$$

$x = 2\omega$

$$= \frac{1}{2}\int_0^1 f(x)\, dx + \frac{1}{2} f(1)$$

$$= \frac{1}{2}\int_0^1 f(x)\, dx + \frac{1}{2}\int_0^1 f(x)\, d\delta_1(x)$$

$$= \int_0^1 f(x)\, d\left(\frac{1}{2}\lambda + \frac{1}{2}\delta_1(x)\right)$$

c)

$X(\omega) = \omega^2 \qquad X : \Omega \to [0,1]$

$B \in \mathcal{B}([0,1])$

$X^{-1}(B) = \sqrt{B} \cup (-\sqrt{B})$

$\sigma(X) = \{ C \cup (-C),\ C \in \mathcal{B}([0,1]) \}$

$\forall f$ continue bornée

$$E[f(X)] = \int_\Omega f(X(\omega))\, dP(\omega) \qquad P = \frac{\lambda}{2}$$

$$= \int_{-1}^{1} f(\omega^2)\, \frac{1}{2}\, d\omega$$

$$= \cancel{2}\int_0^1 f(\omega^2)\, \frac{1}{\cancel{2}}\, d\omega$$

$$= \int_0^1 f(x)\, \frac{1}{2\sqrt{x}}\, dx$$

$$= \int_{\mathbb{R}} f(x) \underbrace{\frac{1}{2\sqrt{x}} \mathbb{1}_{[0,1]}(x)}_{\text{densité}}\, dx$$

Exercice 3

0) $f_{m,\sigma^2}(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

1) $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$E[X] = \int_{\mathbb{R}} x f_X(x)\, dx = 0$

$E[X^2] = \int_{\mathbb{R}} x^2 f_X(x)\, dx$
$\hookrightarrow = 1$

$Var(X) = 1$

$$\int_{-\infty}^{\infty} x^2 e^{-\frac{x^2}{2}}\, dx$$

$$= \int_{-\infty}^{+\infty} (-x)\left(e^{-\frac{x^2}{2}}\right)' dx$$

$$= \left(-x e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} (-x)' e^{-\frac{x^2}{2}}\, dx$$

$$= 0 + \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}}\, dx = \sqrt{2\pi}$$

2) a)

$$m+\sigma X = Y$$

$$\forall f : \mathbb{R} \to \mathbb{R}$$

$$E[f(Y)] = E[f(m+\sigma X)]$$

$$= \int_{\mathbb{R}} f(m+\sigma x)\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} d x$$

$$y = m+\sigma x \iff x = \frac{y-m}{\sigma}$$

$$= \int_{\mathbb{R}} f(y)\frac{1}{\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\sigma^2}} \frac{1}{\sigma} d y$$

densité de Y : $\frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-m)^2}{2\sigma^2}}$

$$Y = m+\sigma X \sim N(m, \sigma^2)$$

b) $E[Y] = E[m+\sigma X] = m + \sigma \underbrace{E[X]}_{=0} = m$

$$Var(Y) = Var(m+\sigma X) = \sigma^2 \underbrace{Var(X)}_{=1} = \sigma^2$$

3) $X, Y \sim N(0,1)$ $\quad Z = \frac{Y}{X}$ $\quad\quad \frac{1}{Z} = \frac{X}{Y}$ m loi

$\forall f$

$$E[f(Z)] = E[f(\tfrac{Y}{X})] = \int_{\mathbb{R}}\int_{\mathbb{R}} f(\tfrac{y}{x}) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dx\, dy$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(\tfrac{y}{x}) e^{\frac{-x^2-y^2}{2}} dx\,dy$$

$$\begin{cases} X = X \\ Z = \frac{Y}{X}\end{cases} \iff \begin{cases} X = X \\ Y = XZ\end{cases} \quad \det\begin{pmatrix} 1 & 0 \\ Z & X\end{pmatrix} = x$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} f(z) e^{\frac{-1+z^2}{2}x^2} |x|\, dx\, dz$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} f(z)\left(\int_{\mathbb{R}} e^{-\frac{1+z^2}{2}x^2}|x|\,dx\right) dz$$

$$= \int_{\mathbb{R}} f(z) \underbrace{\frac{1}{\pi}\frac{1}{1+z^2}}_{\text{densité}} dz$$

4) a) $R = \sqrt{X^2+Y^2}$ $\quad S = \frac{X}{R}$

$\forall f : \mathbb{R}^2 \to \mathbb{R}$ continue bornée

$$E[f(R^2, S)] = E\left[f\left(X^2+Y^2, \frac{X}{\sqrt{X^2+Y^2}}\right)\right]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}_0} f\left(x^2+y^2, \frac{x}{\sqrt{x^2+y^2}}\right) \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} dx\,dy$$

$\to$ pair pour y

$$\begin{cases} z = x^2 + y^2 \\ s = \dfrac{x}{\sqrt{x^2+y^2}} \end{cases} \iff \begin{cases} x = \sqrt{z}\, s \\ y = \sqrt{z}\, \sqrt{1-s^2} \end{cases}$$

$$\det \begin{pmatrix} \dfrac{s}{2\sqrt{z}} & \sqrt{z} \\ \dfrac{\sqrt{1-s^2}}{2\sqrt{z}} & \sqrt{z}\dfrac{(-s)}{\sqrt{1-s^2}} \end{pmatrix} = -\frac{s^2}{2\sqrt{1-s^2}} - \frac{\sqrt{1-s^2}}{2} = \frac{-1}{2\sqrt{1-s^2}}$$

$$\to \; -2\int_0^{+\infty} \int_{-1}^{1} f(z,s)\, \frac{1}{2\pi}$$