Exercice 1

$X \sim Ber\, p$

1) $\phi_X(u) = E[e^{iuX}] = e^{iu1}p + e^{iu0}(1-p) = pe^{iu} + (1-p)$

$p(X=n) = p(1-p)^{n-1} \qquad n \geq 1$

2) $\phi_X(u) = \cancel{\sum_{n\geq 1} e^{iun}(1-p)^{n-1}p} = \sum_{n\geq 1}(1-p)^{n-1}e^{iun}p = \frac{p}{(1-p)}\sum_{n\geq 1}(e^{iu}(1-p))^n$

$= \frac{p}{1-p}\,\frac{e^{iu}(1-p)}{1-e^{iu}(1-p)}$

$= \frac{pe^{iu}}{1-e^{iu}(1-p)}$

3) $\phi_X(u) = \sum_{n\geq 0} e^{iun}\frac{\lambda^n}{n!}e^{-\lambda} = e^{-\lambda}\sum_{n\geq 0}\frac{(e^{iu}\lambda)^n}{n!} = e^{-\lambda}e^{\lambda e^{iu}} = e^{\lambda(e^{iu}-1)}$

4) $f_X(x) = \alpha e^{-\alpha x} \qquad x \geq 0$

$\varphi_X(t) = \int_0^\infty e^{itx}\alpha e^{-\alpha x}dx = \alpha\int_0^\infty e^{(-\alpha+it)x}dx$

Posons $c = -\alpha + it$ $\qquad \frac{1}{c}e^{cx}$

$\varphi_X(t) = \alpha\left(\frac{-1}{-\alpha+it}\right) = \frac{\alpha}{\alpha-it}$

Rq : $\forall c \in \mathbb{C} \quad \int_0^\infty e^{cx}dx = \frac{1}{c}$

5. $X \sim \mathcal{U}([a,b]) \qquad a<b$ sur $\mathbb{R}$

X est à densité sur $\mathbb{R}$ de densité :

$f_X(x) = \mathbb{1}_{[a,b]}(x)\frac{1}{b-a}$ Alors pour $t \in \mathbb{R}$

$\phi_X(t) = \int_a^b e^{itx}f_X(x)dx = \frac{1}{b-a}\int_a^b e^{itx}dx = \frac{1}{b-a}\left[\frac{e^{itx}}{it}\right]_a^b$

$= \frac{1}{b-a}\,\frac{e^{itb}-e^{ita}}{it}$

Exercice 2.

X, Y indépen $\sim Exp(\alpha)$ avec $\alpha > 0$

$\phi_{X-Y}(u) = E[e^{iuX-Y}] = E[e^{iuX}]\,E[e^{iu-Y}] = \phi_X(u)\,\phi_Y(-u)$

$= \frac{\alpha}{\alpha-it}\,\frac{\alpha}{\alpha+it}$

$= \frac{\alpha^2}{\alpha^2+t^2}$

$= \int_0^\infty e^{iux}\alpha e^{-\alpha x}dx$

$\int_0^\infty e^{-iux}\alpha e^{-\alpha x}dx$

$= \int_0^\infty\int_0^\infty e^{iux}e^{-iuy}\alpha e^{-\alpha x}\alpha e^{-\alpha y}dxdy$

$= \int_0^\infty\int_0^\infty e^{iu(x-y)}\alpha^2 e^{-\alpha(x+y)}dxdy$

$$= \int_0^\infty \int_0^\infty e^{iu(x-y)} \alpha^2 e^{-\alpha(x+y)} dx\, dy$$

$$z = x-y \qquad dz = dx \qquad \Leftrightarrow \quad x = z+y$$

$$= \int_0^\infty \int_{-y}^\infty e^{iu(z)} \alpha\alpha e^{-\alpha(z+2y)} dz\, dx \qquad f*g(x) = \int_{-\infty}^\infty f(x) g(y-x) dy$$

$$= \int_{-\infty}^{+\infty} \int_0^z e^{iuz} \alpha\alpha e^{-\alpha(2y)} e^{-\alpha z} dy\, dz$$

$$= \int_{-\infty}^{+\infty} e^{iuz} e^{-\alpha z} \alpha \int_0^z \alpha e^{-\alpha 2y} dy\, dz$$

$$= \int_{-\infty}^{+\infty} e^{iuz} e^{-\alpha z} \alpha \left[-\frac{e^{-\alpha 2y}}{2} \right]_0^z dz$$

$$= \int_{-\infty}^{+\infty} e^{iuz} e^{-\alpha z} \alpha \left(-\frac{e^{-\alpha 2z}}{2} + \frac{1}{2} \right) dz$$

$$= \frac{1}{2} \int_{-\infty}^{+\infty} e^{iuz} \left(1 - \alpha e^{-\alpha 3z} \right) dz$$

$$= \int_{-\infty}^{+\infty} e^{iuz} \frac{1-\alpha e^{-\alpha 3z}}{2} dz$$

$$\int_{\mathbb{R}^2} \mathbb{1}_{\{x\geq 0\}} \alpha e^{-\alpha x} \mathbb{1}_{\{y\geq 0\}} \alpha e^{-\alpha y} g(x-y)\, dx\, dy$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y\geq 0\}} \alpha e^{-\alpha y} \int_{\mathbb{R}} \mathbb{1}_{\{x\geq 0\}} \alpha e^{-\alpha x} g(x-y)\, dx\, dy$$

$$z = x-y \qquad x = z+y$$

$$= \int_{\mathbb{R}} \mathbb{1}_{\{y\geq 0\}} \alpha e^{-\alpha y} \int_{\mathbb{R}} \mathbb{1}_{\{z+y\geq 0\}} \alpha e^{-\alpha(z+y)} g(z)\, dz\, dy$$

$$= \int_{\mathbb{R}} \alpha e^{-\alpha z} g(z) \underbrace{\left(\int_{\mathbb{R}} \mathbb{1}_{\{y\geq 0\}} \mathbb{1}_{\{z+y\geq 0\}} \alpha e^{-2\alpha y} dy \right)}_{\text{si } z\geq 0 \ \int_0^\infty \alpha e^{-2\alpha y} dy = \frac{1}{2} \ ; \ \text{si } z<0 \ \int_{-z}^\infty \alpha e^{-2\alpha y} dy = \frac{1}{2} e^{2\alpha z}} dz$$

$$= \frac{\alpha e^{-\alpha|z|}}{2}$$

la densité de $X-Y$

3. si X a densité f alors $f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi_X(t)\, dt \quad (*)$

Par $(*)$, $f(z) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itz} \varphi_Z(t)\, dt$

i.e. $\frac{\alpha e^{-\alpha|z|}}{2} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \frac{\alpha^2}{\alpha^2+t^2} dt \quad (**)$

Si $W \sim \text{Cauchy}(\lambda)$, W à densité $f_W(x) = \frac{\lambda}{\pi(\lambda^2+x^2)}$

$\varphi_W(x) =$

On prend (**) avec :

$\xi = -u \qquad \alpha = \lambda$

$$\frac{\lambda e^{-\lambda|u|}}{2} = \frac{1}{2}\lambda \varphi_W(u), \text{ donc } \quad \varphi_W(u) = e^{-\lambda|u|} \qquad u \in \mathbb{R}.$$

4.

$U \sim \text{Cauchy}(\lambda) \qquad V \sim \text{Cauchy}(\mu)$

$\varphi_U(t) = e^{-\lambda|t|} \qquad \varphi_V(t)\ e^{-\mu|t|}$

$$\varphi_{\alpha U+\beta V}(t) = \varphi_{\alpha U}(t)\,\varphi_{\beta V}(t) = \varphi_U(\alpha t)\,\varphi_V(\beta t) = e^{-\lambda|\alpha t|}\, e^{-\mu|\beta t|} = e^{-(\alpha\lambda+\beta\mu)|t|}$$

$$\varphi_{\alpha U}(t) = E[e^{it\alpha U}] = \int_{\mathbb{R}} e^{it\alpha u} f(u)\,dt = E[e^{i(\alpha t)u}] = \varphi_U(\alpha t)$$

Donc $U+V$ est une variable aléatoire $\sim \text{Cauchy}(\alpha\lambda+\beta\mu)$ car la fonction caractéristique caractérise la loi.

Exercice 3

1) X, Y indépts

$$\varphi_{X+Y}(u) = E[e^{iu(X+Y)}] = E[e^{iuX} e^{iuY}] = E[e^{iuX}]E[e^{iuY}] = \varphi_X(u)\varphi_Y(u)$$

2) $\varphi_{X_1+\dots+X_n}(u) = \varphi_{X_1}(u)\varphi_{X_2}(u)\dots\varphi_{X_n}(u)$ par récurrence et 1)

3) $\varphi_{B(n,p)}(u) = (pe^{iu} + (1-p))^n$

4) $\varphi_{X_1+\dots+X_n}(u) = e^{(e^{iu}-1)(\lambda_1+\dots+\lambda_n)}$

5) $\varphi_{X_1+\dots+X_n}(u) = \left(\frac{\lambda_1}{\lambda_1 - it}\right)\left(\frac{\lambda_2}{\lambda_2 - it}\right)\dots\left(\frac{\lambda_n}{\lambda_n - it}\right)$

6) $\varphi_{\mathcal{N}(m_i;\sigma_i^2)}(u) = e^{ium_i - \frac{1}{2}\sigma_i^2 u^2}$

$$\varphi_{\sum \mathcal{N}(m_i,\sigma_i^2)}(u) = e^{\sum_i (ium_i - \frac{1}{2}\sigma_i^2 u^2)}$$