mode d'une v.a =

le x tq $P(X=x)$ soit maximale
s'il en existe

Exercice 1

1) Si $Y_1, \dots, Y_n \sim$ Bernoulli(p)

$P(Y_i=0) = 1-p$ $\quad$ $P(Y_i=1) = p$

Alors $\tilde{S}_n = Y_1 + \dots + Y_n \sim \mathcal{B}in(n,p)$

$P(\tilde{S}_n = k) = \binom{n}{k} p^k (1-p)^{n-k}$

S_n a la même loi que $2\tilde{S}_n - n = \sum (2Y_i - 1)$

X_i a la même loi que $\underline{2Y_i - 1}$

$\frac{S_n + n}{2} \sim \mathcal{B}in(n,p)$

$$P(S_n = k) = P\left(\tilde{S}_n = \frac{n+k}{2}\right) = \begin{cases} 0 & \text{si } \frac{n+k}{2} \notin \mathbb{Z} \\ \binom{n}{\frac{n+k}{2}} p^{\frac{n+k}{2}} (1-p)^{\frac{n+k}{2}} & \text{sinon} \end{cases}$$

$p^n (1-p)^n$ $\qquad$ $\binom{n}{\frac{n}{2}} p^{\frac{n}{2}} (1-p)^{\frac{n}{2}}$

$C_n^{\frac{n}{2}}$

plus formellement:

$$P(S_n = k) = P\left(\sum_i^n X_i = k\right) = \sum_{\substack{x_1,\dots,x_n \\ \text{tq } \sum_i x_n = k}} P(X_1 = x_1 \cap \dots \cap X_n = x_n)$$

$$= \sum_{\substack{x_1,\dots,x_n \\ \text{tq } \sum_i x_n = k}} P(X_1 = x_1) \cdot \ldots \cdot P(X_n = x_n)$$

$$= \sum_{\substack{x_1,\dots,x_n \\ \text{tq } \sum_i x_n = k}} P(2Y_1 - 1 = x_1) \cdot \ldots \cdot P(2Y_n - 1 = x_n)$$

3) $E[S_n] = 2np - n = n(2p-1)$

4) $Var(2\tilde{S}_n - n) = 4\, Var(\tilde{S}_n) = 4np(1-p)$

$\parallel$

$$Var(S_n) = \sum_k (k - (2p-1)n)^2 P(S_n = k)$$

2) Pour quel k $P(S_n = k)$ est max

Pour quel k a-t-on $p_{k+1} > p_k$

$$q_k = \binom{n}{k} p^k (1-p)^{n-k}$$

Soit $0 \le k \le n$, on a

$$q_{k+1} > q_k \iff \frac{q_{k+1}}{q_k} > 1 \iff \frac{\frac{n!}{(k+1)!\,(n-k-1)!}}{\frac{n!}{k!\,(n-k)!}} \frac{p^{k+1}}{p^k} \frac{(1-p)^{n-k-1}}{(1-p)^{n-k}} > 1$$

$$\iff \frac{\cancel{n!}}{\cancel{n!}} \frac{k!}{(k+1)!} \frac{(n-k)!}{(n-k-1)!} \frac{p}{1-p} > 1$$

$$\iff \frac{n-k}{k+1} \frac{p}{1-p} > 1$$

$$\iff np - kp > k+1 - (k+1)p$$

$$\iff (n+1)p > k+1$$

$$K_{max} = \lfloor (n+1)p \rfloor - 1$$

car $\underline{q_{k+1} > q_k}$

$q_0 < q_1 < \dots < q_{k_{max}} < q_{k_{max}+1}$

$q_{k_{max}+1} > q_{k_{max}+2} > \dots > q_n$

Donc le mode de S_n est $k_{max}+1 = \lfloor (n+1)p \rfloor$

5) $n! \sim \sqrt{2n\pi}\left(\frac{n}{e}\right)^n$

6) Inégalité de Bienaymé-Chebychev

5) $P(S_{2n}=0) = \binom{2n}{\frac{2n}{2}} p^{\frac{2n}{2}} (1-p)^{\frac{2n}{2}}$

$= \binom{2n}{n} p^n (1-p)^n$

$= \frac{2n!}{(2n-n)!\,n!} p^n (1-p)^n = \frac{(2n)!}{n!\,n!} (1-p)^n p^n$

$= \frac{\cancel{2}\sqrt{n\pi}\left(\frac{2n}{e}\right)^{2n}}{\cancel{2}n\pi\left(\frac{n}{e}\right)^{2n}} p^n (1-p)^n$

$= \frac{\sqrt{n\pi}\,2^{2n}}{n\pi} \underbrace{p^n (1-p)^n}_{\xrightarrow[n\to+\infty]{} 0 \text{ et } p \neq \frac{1}{2}} = \frac{4^n}{\sqrt{n\pi}}$

Si $p = \frac{1}{2}$ $\quad \frac{1}{\sqrt{n\pi}} 2^{2n} \left(\frac{1}{2}\right)^{2n} = \frac{1}{\sqrt{n\pi}} \xrightarrow[n\to+\infty]{} 0$

si $p \neq \frac{1}{2}$

6) $S_n = 2\tilde{S}_n - n$ $\qquad E[S_n] = 2np - n$

$$P(|S_n - np| > t)$$
$$\parallel$$
$$P(|2\tilde{S}_n - n - 2np + n| > t)$$
$$= P(|2\tilde{S}_n - 2np| > t)$$
$$= P(|\tilde{S}_n - np| > \tfrac{t}{2})$$
$$= P(|\tilde{S}_n - E[\tilde{S}_n]| > \tfrac{t}{2}) < \frac{Var(\tilde{S}_n)}{\frac{t^2}{4}}$$
$$= \frac{4Var(\tilde{S}_n)}{t^2}$$
$$= \frac{4np(1-p)}{t^2}$$

Exercice 2

$p \in]0,1[$ $\quad (X_n)_{n\geq 1}$

$\forall n \geq 1, \quad P(X_n = -1) = 1-p \qquad P(X_n = 1) = p$

$T = \min\{n \geq 1 : X_n = 1\}$

1. T décrit la première i de X_i quand $X_i = 1$

Pour $i \geq 1$

$P(T = i) = (1-p)^{i-1} p$

La loi s'appelle la loi géométrique

2) $E[T] = \sum_{i \in \mathbb{N}^*} i \, P(T = i) = \sum_{i \in \mathbb{N}^*} i \, (1-p)^{i-1} p$

$= \frac{1}{p}$

3) $Var(T) = E[T^2] - \underbrace{E[T]^2}_{= \frac{1}{p^2}} = \frac{1-p}{p^2}$

$E[T^2] = \sum_{i \in \mathbb{N}} i^2 P(T = i) = \sum_{i \in \mathbb{N}} i^2 p (1-p)^{i-1}$

Exercice 3

1) $R \sim \mathcal{H}(N, K, n)$

$$\forall k \in [0, n] \; P(R = k) = \frac{\binom{K}{n}\binom{N-K}{n-k}}{\binom{N}{n}}$$

Hypergéométrique

2) $\frac{nK}{N}$

3) $\frac{N-n}{N-1} \cdot \frac{nK}{N}\left(1 - \frac{K}{N}\right)$

4) $P(T = n) = \frac{\binom{n-1}{0}\binom{N-n}{K-1}}{\binom{N}{K}}$

négative hypergéométrique

5) $\frac{r\,N-1}{K-1} = E[T]$

6) $Var(T) = \frac{r(M-r+1)(N+1)(N-M)}{(M+1)^2(M+2)}$

Exercice 4

$\alpha \in \mathbb{R}^*$

$n \in \mathbb{N}^*$ $\quad (X_k^n)_{1 \leq k \leq n}$ $\quad \forall k \quad X_k^n \in \mathcal{B}(\frac{\alpha}{n})$

$$S_n = X_1^n + \dots + X_n^n$$

1) $P(S_n = x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left(\frac{\alpha}{n}\right)^x \left(1 - \frac{\alpha}{n}\right)^{n-x}$

On a $S_n \sim \mathcal{B}in(n, \frac{\alpha}{n})$ quand $n \to +\infty$
d'après le théorème du cours

$$S_n \sim \mathcal{P}(n \frac{\alpha}{n}) = \mathcal{P}(\alpha)$$

2) $E[S_n] \xrightarrow[n \to +\infty]{} \alpha$

$Var(S_n) \xrightarrow[n \to +\infty]{} \alpha$

Exercice 5

$(X_k)_{1 \leq k \leq n}$ i.d.

$$S_n = S_1 + \dots + S_n$$

Supposons que $E(X_1^2) < +\infty$

1) $E[X_1^2] = \sum_{x \in X(\Omega)} x^2 P(X_1 = x) < +\infty$

i.e $\exists \ell \in \mathbb{R}$ tq $\sum_{x \in X(\Omega)} x^2 P(X_1 = x)$

$\forall x \in]-\infty, -1] \cup [1, +\infty[$, $|x| \leq x^2$

$(|x| - 1)^2 = x^2 - 2|x| + 1 \geq 0 \Rightarrow |x| \leq \frac{1 + x^2}{2}$

alors $E[|X|] \leq \frac{1 + E[X^2]}{2}$ avec $E[X^2] < +\infty$ donnent

$E[|X|] < +\infty$

2] Si X est de carré intégrable.

$$P(|X - E[X]| > \varepsilon) < \frac{Var(X)}{\varepsilon^2}$$

3] $E[S_n] = n\,E[X_1]$

$$Var(S_n) = E[S_n^2] + E[S_n]^2$$

$$= E[(n X_1)^2] + n^2 E[X_1]^2$$

$$= n^2 \left(E[X_1^2] + E[X_1]^2\right)$$

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