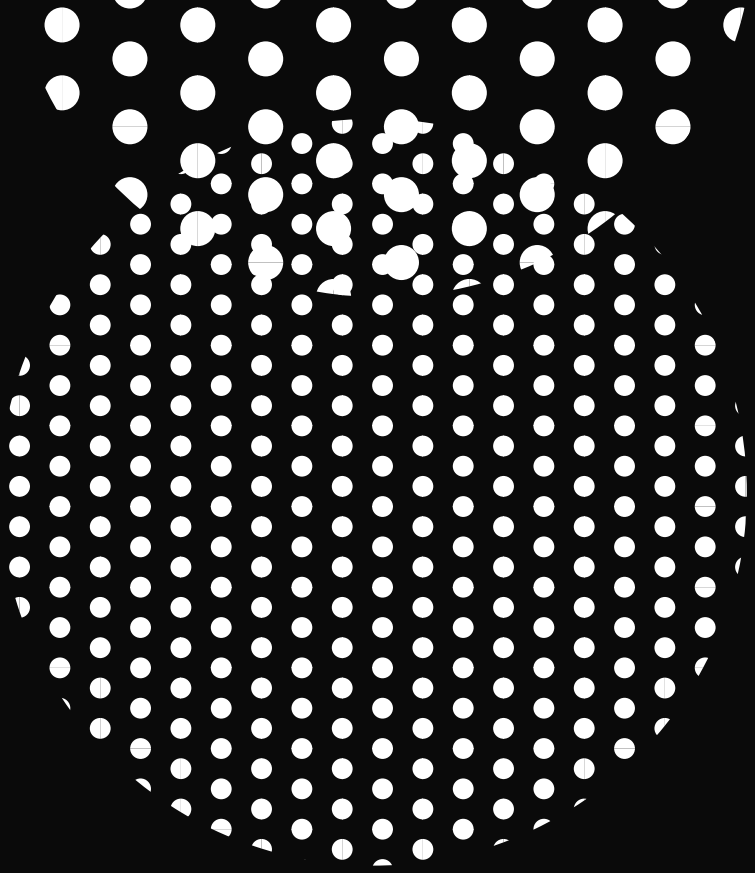


Exercice 1

Supposons que X_n cv. vers X en loi, donc

$\forall g$ continue bornée

$$\lim_{n\to+\infty} E[g(X_n)] = E[g(X)]$$

$$\Leftrightarrow \lim_{n\to+\infty} \int_{\mathbb{R}} f(X_n)\,dP = \int_{\mathbb{R}} f(X)\,dP$$

Soit f continue, or g bornée, donc $g\circ f$ continue bornée.

d'où

$$\lim_{n\to+\infty} E[g(f(X_n))] = E[g\circ f(X)]$$

Donc en notant $Y_n := f(X_n)$ et $Y := f(X)$, on a

$$\lim_{n\to+\infty} E[g(Y_n)] = E[g(Y)]$$ donc $f(X_n)$ converge en loi vers $f(X)$.

Exercice 2

Soit $n\in\mathbb{N}$, soit $x\in\mathbb{R}$

$$F_{U_n}(x) = P(X_1\le x \cup X_2\le x \cup \dots \cup X_n \le x)$$

$$1 - F_{U_n}(x) = 1 - P(X_1\le x \cup X_2\le x \cup \dots \cup X_n \le x)$$

$$= P((X_1\le x \cup X_2\le x \cup \dots \cup X_n \le x)^c)$$

$$= P(X_1>x \cap X_2>x \cap \dots \cap X_n>x)$$

$$= (1-F_{X_1}(x)) \dots (1-F_n(x))$$

$$F_{U_n}(x) = 1 - (1-F_{X_1}(x)) \dots (1-F_n(x))$$

$$F_{V_n}(x) = P(X_1\le x \cap \dots \cap X_n\le x) = P(X_1\le x)\dots P(X_n\le x)$$
$$= F_{X_1}(x)\dots F_{X_n}(x)$$

2.

$$F_{U_n}(x) = 1 - (1-F_{X_1}(x))^n$$

$$F_{V_n}(x) = (F_{X_1}(x))^n$$

3. Si $X_1 \sim \mathrm{Unif}([0,1])$ On a $F_{X_1}(x) = \begin{cases} 0 & \text{si } x\le 0 \\ x & \text{si } x\in]0,1[\\ 1 & \text{si } x\ge 1 \end{cases}$

Donc

$$F_{U_n}(x) = 1-(1-F_1(x))^n \xrightarrow[n\to+\infty]{} \begin{cases} 0 & \text{si } x\le 0 \\ 1 & \text{si } x>0 \end{cases} = \mathbb{1}_{]0,+\infty[}$$

$$F_{V_n}(x) = (F_{X_1}(x))^n \xrightarrow[n\to+\infty]{} \begin{cases} 0 & \text{si } x<1 \\ 1 & \text{si } x\ge 1 \end{cases} = \mathbb{1}_{[1,+\infty[}$$

(6) Si $X_1 \sim Exp(\lambda)$

$F_{X_1}(x) = 1 - e^{-\lambda x}$

$F_{U_n}(x) = 1 - [1 - F_{X_1}(x)]^n = 1 - (1 - (1 - e^{-\lambda x}))^n = 1 - e^{-\lambda x n} \xrightarrow[n \to +\infty]{} 1$

$F_{V_n}(x) = (F_{X_1}(x))^n = (1 - e^{-\lambda x})^n$

$1 - e^{-\lambda x} = 0$ si $x = 0$, si $x \neq 0$, si $x < 0$, $F_{X_1}(x) = 0$
si $x > 0$ $F_{X_1}(x) = 1 - e^{-\lambda x}$

Or $\forall x > 0$ $e^{-\lambda x} > 0$ et $e^{-\lambda x} < 1$ donc $1 - e^{-\lambda x} \in]0, 1[$

Donc $F_{U_n}(x) \xrightarrow[n \to +\infty]{} \begin{cases} 0 \text{ si } x > 0 \\ 1 \text{ si } x \leq 0 \end{cases}$

$F_{V_n}(x) \xrightarrow[n \to +\infty]{} 0$

Exercice 3

1. X v.a $X = \int_0^X 1 \, dt$

$$\begin{aligned} E[X] = \int_{\mathbb{R}_+} X \, dP_X &= \int_0^{+\infty} x f(x) \, dx \\ &= \int_0^{+\infty} x \, dP_X \\ &= \int_0^{+\infty} \left(\int_0^x 1 \, dt \right) dP_X \\ &= \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{\{t \leq x\}} \, dt \, dP_X \\ &= \int_0^{+\infty} \left(\int_0^{+\infty} \mathbb{1}_{\{x \geq t\}} \, dP_X \right) dt \\ &= \int_0^{+\infty} 1 - F_X(t) \, dt \end{aligned}$$

2. $E[X] = \sum_{n \in \mathbb{N}} n P(X = n)$

$$\sum_{n \in \mathbb{N}} 1 - F_X(n) = \sum_{k=0}^{+\infty} P(X \geq k) = \sum_{k=0}^{+\infty} \sum_{j=k}^{+\infty} P(X = j)$$

$$= \sum_{j=0}^{+\infty} \sum_{k=0}^{j} P(X = j) = \sum_{j=0}^{+\infty} j P(X = j) = E[X]$$

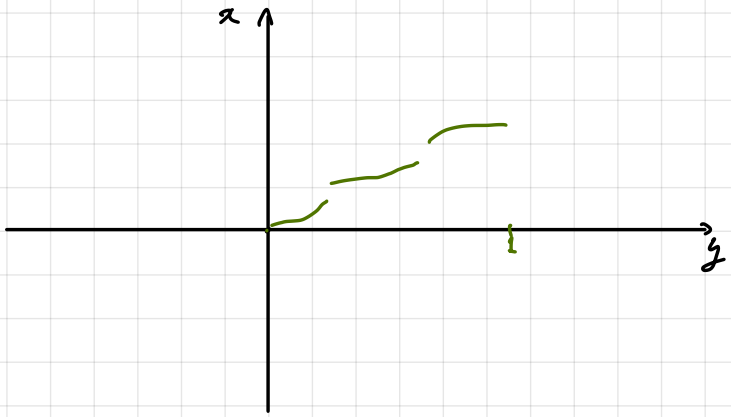
3] $X^p = \int_0^X p t^{p-1} dt$

$$E[X^p] = \int_0^{+\infty} x^p \, dP_X$$
$$= \int_0^{+\infty} \left(\int_0^x p t^{p-1} dt \right) P_X$$
$$= \int_0^{+\infty} \left(\int_0^{+\infty} \mathbb{1}_{\{t \le x\}} p t^{p-1} dt \right) P_X$$
$$= \int_0^{+\infty} \left(\int_0^{+\infty} \mathbb{1}_{\{x \ge t\}} p t^{p-1} dP_X \right) dt$$
$$= \int_0^{+\infty} p t^{p-1} (1 - F_X(t)) \, dt$$

Exercice 4 $\quad F(x) = P[X \le x]$

Soit $y \in [0,1]$, $\quad G(y) = \inf\{x \in \mathbb{R} : F(x) \ge y\}$

1.



2. Supposons $X \sim Unif([0,1])$

$$F(x) = \mathbb{1}_{[0,1]}(x) \, x$$

$$G(y) = y$$

Supposons $X \sim Exp(\alpha)$

$$F(x) = 1 - e^{-\alpha x}$$

$$G(y) = -\frac{\log(1-y)}{\alpha}$$

$$1 - e^{-\alpha x} \ge y \iff e^{-\alpha x} \le 1 - y$$
$$\iff -\alpha x \le \log(1-y)$$
$$\iff x \ge \frac{-\log(1-y)}{\alpha}$$

3. $f(u)$

$$E[f(G(u))] = \int f(G(u)) \, \mathbb{1}_{[0,1]} \, dP_U$$
$$= \int f(u) \, \mathbb{1}_{[0,1]} \, dP_U$$

4. Si $Y = F(X)$, or F est strictement croissante et continue donc elle admet une vraie inverse.

$P(Y \le y) = P(F(X) \le y) = P(X \le F^{-1}(y)) = F(F^{-1}(y))$ donc $F(X)$ suit $Unif([0,1])$

Exercice 5

1. $(X_n)_{n\in\mathbb{N}}$ converge en loi vers X i.e

$$E[f(X_n)] \xrightarrow[n\to+\infty]{} E[f(X)]$$

i.e $\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n \geq N \quad \left| \int_0^{+\infty} f(x_n) P_{X_n} - \int_0^{+\infty} f(x) d P_X \right| < \varepsilon$

$$E[f(X_n) - f(X)] \leq \varepsilon$$

2. F est continue en tout point donc $\forall x \in \mathbb{R}$

$$\lim_{n\to+\infty} F_{X_n}(x) = F_X(x)$$

Exercice 1

Supposons que $X_n \xrightarrow{\mathcal{L}} X$ i.e $\forall g$ continue bornée

$$\lim_{n\to\infty} E[g(X_n)] = E[g(X)]$$

Or f est continue bornée donc $g\circ f$ est continue bornée d'où

$$\lim_{n\to\infty} E[g\circ f(X_n)] = E[g\circ f(X)]$$

$$\Downarrow$$

$$E[g(f(X_n))] = E[g(f(X))]$$

d'où $f(X_n) \xrightarrow{\mathcal{L}} f(X)$

Exercice 2

1. Soit $t \in \mathbb{R}$

$$F_{V_n}(t) = P(V_n \le t) = P(X_1 \le t \wedge \dots \wedge X_n \le t) = P(X_1 \le t)\dots P(X_n \le t) \quad \text{car indépendes.}$$

$$= F_{X_1}(t)\dots F_{X_n}(t)$$

$$F_{U_n}(t) = P(U_n \le t) = 1 - P(U_n > t)$$

$$P(U_n > t) = P(X_1 > t \wedge \dots \wedge X_n > t)$$

$$= P(X_1 > t)\dots P(X_n > t)$$

$$= (1 - F_{X_1}(t))\dots(1 - F_{X_n}(t))$$

d'où $F_{U_n}(t) = 1 - ((1-F_{X_1}(t))\dots(1-F_{X_n}(t)))$

2. $F_{V_n}(t) = F_{X_1}(t)^n$

$$F_{U_n}(t) = 1 - (1-F_{X_1}(t))^n$$

3. Unif $F_{V_n}(t) = t^n \longrightarrow \begin{cases} 0 & \text{si } t < 1 \\ 1 & \text{si } t \ge 1 \end{cases}$ converge vers 1

$F_{U_n}(t) = 1 - (1-t)^n \longrightarrow \begin{cases} 0 & \text{si } t \le 0 \\ 1 & \text{si } t > 0 \end{cases}$ converge vers 0.

6) exp.

$$F_{V_n}(t) = \left(1 - e^{-\lambda t}\right)^n \xrightarrow[n \to +\infty]{} 0 \quad \forall t$$

qui n'est pas correcte donc V_n ne converge pas en loi.

$$F_{U_n}(t) = 1 - \left[1 - (1 - e^{-\lambda t})\right]^n$$

$$= 1 - \left(e^{-\lambda t}\right)^n \xrightarrow[n \to +\infty]{} \begin{cases} 0 & \text{si } t \leq 0 \\ 1 & \text{si } t > 0 \end{cases}$$

converge vers 0.

Exercice 3

1. $E[X] = \int_0^{+\infty} x \, dP_X$

$$= \int_0^{+\infty} x f_X(x) \, dx$$

$$= \int_0^{+\infty} f_X(x) \left(\int_0^x t \, dt\right) dx$$

$$= \int_0^{+\infty} \left(\int_t^{+\infty} t f_X(x) \, dx\right) dt$$

$$= \int_0^{+\infty} t \int_t^{+\infty} f_X(x) \, dx \, dt$$

$$= \int_0^{+\infty} t \left(1 - F_X(t)\right) dt$$

2.

$$E[X] = \sum_{n \in \mathbb{N}} n P(X = n)$$

$$= \sum_{n \in \mathbb{N}} \left(\sum_{k=1}^{n} 1\right) P(X = n)$$

$$= \sum_{k=1}^{+\infty} \sum_{n=k}^{+\infty} P(X = n)$$

$$= \sum_{k=1}^{+\infty} 1 - F_X(n)$$

3. $p > 0$ $\quad X^p = \int_0^x p t^{p-1} \, dt$

$$E[X^p] = \int_0^{+\infty} x^p f_X($$