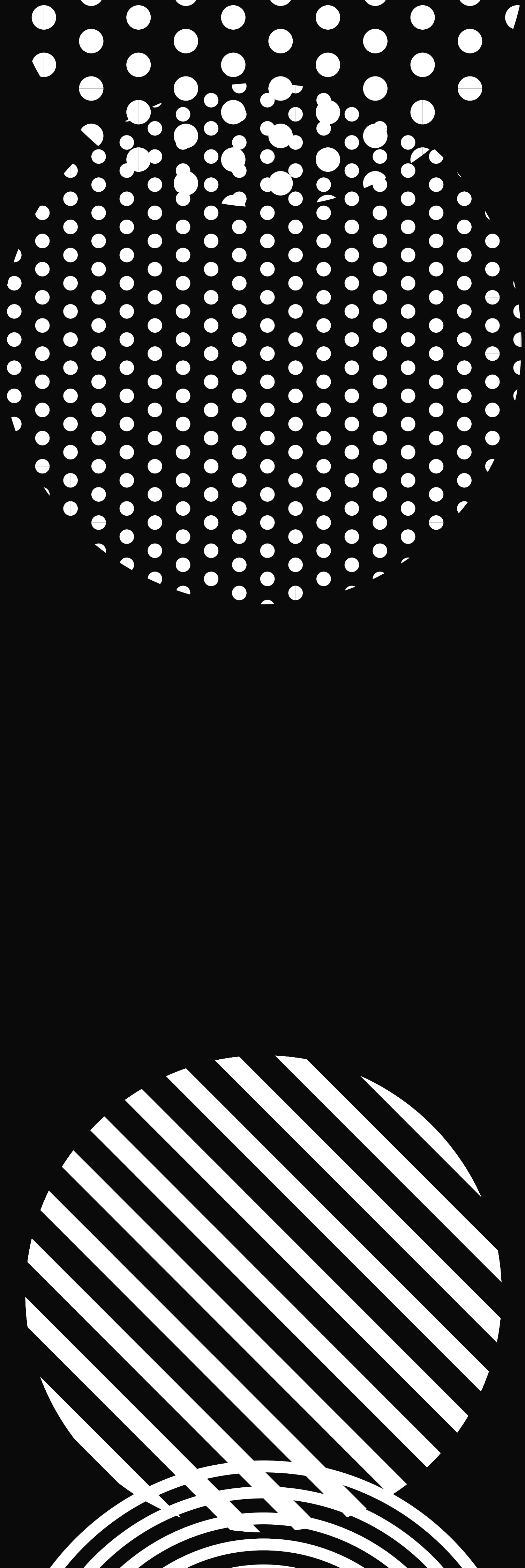


Exercice 1

$X \sim \mathcal{U}nif([0,1])$

0) $\left([0,1],\ \mathcal{B}([0,1]),\ \dfrac{Leb(\cdot \cap [0,1])}{Leb([0,1])}\right)$

1) $\mu(X \in \mathbb{Q}) = \sum_{x \in \mathbb{Q}} \underbrace{\mu(X = x)}_{=0} = 0$

car $\mathbb{Q}$ union disjointe des singletons les rationnels.

2) Soit $(X_0, X_1, \dots)$ v.a qui décrit l'écriture de X. $\forall i \in \mathbb{N}$ $X_i \sim \mathcal{U}nif(\{0,\dots,9\})$

i.e $\forall q \in \{0,\dots,9\}$ $P(X_i = q) = \frac{1}{10}$

Montrons que X_i sont indépendantes.

$$X = \frac{X_0}{10} + \frac{1}{10} \cdot 10X - \frac{X_0}{10} = \frac{X_0}{10} + \frac{1}{10}\underbrace{(10X - X_0)}_{X'}$$

$P(X \in [a,b[) = \lambda([a,b[)$

$$\begin{aligned}
P(X' \in [a,b[\mid X_0 = k) &= P(X' \in [a,b[\cap X_0 = k) / P(X_0 = k) \\
&= P(10X - k \in [a,b[\ ) / P(X_0 = k) \\
&= P(X \in [\tfrac{a}{10} + k, \tfrac{b}{10} + k[) / P(X_0 = k) \\
&= \frac{\frac{b-a}{10}}{\frac{1}{10}} = b - a
\end{aligned}$$

D'où $P(X_0 = k \cap X' \in [a,b[) = P(X' \in [a,b[\mid X_0 = k) P(X_0 = k)$
$= \frac{b-a}{10}$

$$\begin{aligned}
P(X' \in [a,b[) &= \sum_{k=0}^{9} P(X_0 = k) P(X' \in [a,b[\mid X_0 = k) \\
&= \sum_{k=0}^{9} \frac{b-a}{10} = 10 \cdot \frac{1}{10}\, b-a = b-a
\end{aligned}$$

donc $X' \sim \mathcal{U}nif([0,1])$ et X' et X_0 indep.

D'où $P(X \text{ n'a pas de } 7) = P(X_0 \neq 7)\, P(X' \text{ n'a pas de } 7)$

X et X' suivent la même loi

donc $p := P(X \text{ n'a pas de } 7) = P(X' \text{ n'a pas de } 7)$

d'où $p = \frac{9}{10} p \Rightarrow p = 0$

Par complémentaire $P(X \text{ a } 7) = 1 - P(X \text{ n'a pas de } 7)$

$= 1 - 0 = \underline{1}$

3) $Y = \pi(X - \frac{1}{2}) = f(X)$

$$\boxed{\int f\, d\mu = \int f\, d\lambda \quad \text{alors} \quad \mu = \lambda}$$

$\int g(y)\, dP_Y = \int$

$\shortparallel$

$E[g(Y)] = E[g \circ f(X)] = \int f(X)\, dP_X$

$\shortparallel$

$\int g \circ f\, dP_X = \int_0^1 g \circ f(x)\, dP_X(x)$

$= \int_0^1 g(\pi(x - \frac{1}{2}))\, dP_X(x) = \int_0^1 g(\pi(x - \frac{1}{2}))\, \underbrace{h(x)}_{\mathbb{1}_{[0,1]}(x)}\, dx$

$y_0 = \pi(0 - \frac{1}{2}) \Rightarrow y_0 = -\frac{\pi}{2}$

$y_1 = \pi(1 - \frac{1}{2}) \Rightarrow y_1 = \frac{\pi}{2}$

$y = \pi(x - \frac{1}{2}) \Rightarrow x = \frac{y}{\pi} + \frac{1}{2}$

$\frac{dy}{dx} = \frac{dy}{dx} \pi(x - \frac{1}{2}) = \pi \Rightarrow dy = \pi\, dx \Rightarrow dx = \frac{dy}{\pi}$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(y) \underbrace{\frac{1}{\pi} h\left(\frac{y}{\pi} + \frac{1}{2}\right)}_{P_Y} dy$

$h_X = \mathbb{1}_{[0,1]}(x)$

$\frac{y}{\pi} + \frac{1}{2} = 0 \Rightarrow -\frac{\pi}{2}$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(y) \frac{\mathbb{1}_Y(y)}{\pi}\, dy$

$\frac{\mathbb{1}_Y(y)}{\pi} = P_Y(y)$

donc $Y \sim \mathcal{U}nif([-\frac{\pi}{2}, \frac{\pi}{2}])$

$h_X(\frac{y}{\pi} + \frac{1}{2}) = \mathbb{1}_{[-\frac{\pi}{2}, \frac{\pi}{2}]}(y)$

4) $Q = \tan Y$

$$\int g(Q)\, dP_Q = \int g \circ \tan(Y)\, dP_Y$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} g(\tan(y))\, P_Y(y)\, dy$$

СУКА НЕУДАЧА

$u_0 = \tan(\frac{\pi}{2}) = \frac{\sin}{\cos} = -\infty$

$u_1 = +\infty$

$u = \tan(y) \Rightarrow y = \arctan(u)$

$\frac{du}{dy} = \frac{1}{\cos^2(y)} \Rightarrow dy = \cos^2(y)\, du = \cos^2(\arctan(u))\, dy$

$$\int_{-\infty}^{+\infty} g(u) \cos^2(\arctan(u))\, f_Y(\arctan(u))\, du$$

$$\int g(Q)\, dP_Q = \int_0^1 g(\tan(\pi(x - \tfrac{1}{2})))\, \underset{\mathbb{1}_{[0,1]}(x)}{\underset{\shortparallel}{h(x)}}\, dx$$

$=$

$u = \tan(\pi(x - \frac{1}{2}))$ $\Rightarrow$ $\arctan(u) = \pi(x - \frac{1}{2})$

$\Rightarrow \frac{\arctan(u)}{\pi} = x - \frac{1}{2}$

$\Rightarrow \frac{\arctan(u)}{\pi} + \frac{1}{2} = x$

$\sin(\pi x - \frac{\pi}{2})$

$\frac{dx}{du} = \frac{1}{\pi} \frac{1}{1+u^2} = \frac{1}{\pi(1+u^2)}$

$\Rightarrow dx = \frac{1}{\pi(1+u^2)}\, du$

$$\int_{\mathbb{R}} f(u)\, \underset{1}{\underset{\shortparallel}{h(x)}}\, \frac{1}{\pi(1+u^2)}\, du = \int_{\mathbb{R}} f(u) \frac{1}{\pi(1+u^2)}\, du$$

$h(\frac{\arctan(u)}{\pi} + \frac{1}{2})$

$\Leftrightarrow \quad -\frac{\pi}{2} \le \arctan(u) \le \frac{\pi}{2}$

$\Leftrightarrow \quad -\frac{1}{2} \le \frac{\arctan(u)}{\pi} \le \frac{1}{2}$

$\Leftrightarrow \quad 0 \le \frac{\arctan(u)}{\pi} + \frac{1}{2} \le 1$

Donc $\tan(Y)$ suit la loi de densité $u \longmapsto \frac{1}{\pi(1+u^2)}$

6)

$$E[|\tan(Y)|] = \int_{\mathbb{R}} \frac{|u|}{\pi(1+u^2)}\, du$$

$$= \frac{2}{\pi} \int_0^{+\infty} \frac{u}{1+u^2}\, du$$

$t \quad 1+u^2$ $\qquad t_0 = +\infty$, $t_1 = 1$

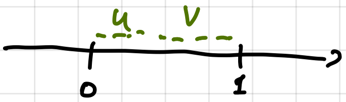
$$\frac{dt}{du} = 2u$$

$$\Rightarrow dt = 2u\,du \Rightarrow du = \frac{dt}{2u}$$

$$\frac{1}{\pi}\int_0^{+\infty} \frac{1}{t}\, dt = \frac{1}{\pi}\left[\ln(t)\right]_1^{+\infty} = \frac{1}{\pi}(\ln(+\infty)) - 0 \longrightarrow +\infty$$

Donc $E[|\tan(Y)|]$ n'existe pas.

Exercice 2



$$\mathcal{U} = \min(X, 1-X)$$

$$V = \max(X, 1-X)$$

Pour $u \in [0, \frac{1}{2}]$

$$F_{\mathcal{U}}(u) = P(\mathcal{U} \leq u) = 1 - P(\mathcal{U} > u) = 1 - P(X \in]u, 1-u[)$$
$$= 1 - (1-2u) \quad \text{car } X \sim Unif([0,1])$$

$$\mathcal{U} > u \iff \begin{cases} X > u & \Rightarrow u < X \\ 1 - X > u & \Rightarrow X < 1-u \end{cases} \Rightarrow u < x < 1-u$$

d'où $F_{\mathcal{U}}(u) = 2u$ pour $u \in [0, \frac{1}{2}]$

et $\frac{dF_{\mathcal{U}}}{du} = 2 = \frac{1}{\frac{1}{2} - 0} = \frac{1}{\frac{1}{2}} = 2$

donc $\mathcal{U} \sim Unif([0, \frac{1}{2}])$

$V = \max(X, 1-X)$

pour $v \in [\frac{1}{2}, 1]$

$F_V(v) = P(V \leq v) = P(X \in [1-v, v]) = v - (1-v) = 2v + 1$

$\Updownarrow$

$\begin{cases} X \leq v \\ 1 - X \leq v \end{cases} \Rightarrow 1 - v \leq X \qquad 1-v \leq X \leq v$

pour $v \in]\frac{1}{2}, 1[$

$\frac{dF_V}{dv}(v) = 2 = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2$

donc $V \sim \mathcal{U}nif([\frac{1}{2}, 1])$

2) $E[\mathcal{U}] = \int_0^{\frac{1}{2}} u \cdot 2\, du = 2\left[\frac{1}{2}u^2\right]_0^{\frac{1}{2}} = 2 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{4}$

$E[V] = \int_{\frac{1}{2}}^{1} u\, 2du = 2\left[\frac{1}{2}u^2\right]_{\frac{1}{2}}^{1} = 2 \cdot \frac{1}{2}\left(1 - \frac{1}{4}\right) = \frac{3}{4}$

$E[V]/E[\mathcal{U}] = \frac{\frac{3}{4}}{\frac{1}{4}} = 3$

$x \in [1, +\infty[$

$P(\frac{V}{\mathcal{U}} < x) = P(V < x\mathcal{U}) = P(V < x\mathcal{U} \mid \mathcal{U} = u)\, P(\mathcal{U} = u)$

$V = 1 - \mathcal{U}$ donc $Y = \frac{1-\mathcal{U}}{\mathcal{U}}$

$E[g(Y)] = \int g\, dP_Y = \int g\left(\frac{1-u}{u}\right) dP_{\mathcal{U}}$

$= \int g \circ f(u)\, \underbrace{h(u)}_{=2} du$

$f(x) = \frac{1-x}{x}$

$= \int_0^{\frac{1}{2}} g \circ f(u) \cdot 2\, du = \int_0^{\frac{1}{2}} g\left(\frac{1-u}{u}\right) 2\, du$

$y = \frac{1-u}{u} = \frac{1}{u} - 1 \qquad y_0 = +\infty$

$\qquad y_1 = 1$

$\Leftrightarrow y + 1 = \frac{1}{u}$

$\Leftrightarrow u = \frac{1}{y+1}$

$\Leftrightarrow \frac{du}{dy} = -\frac{1}{(y+1)^2}$

$\Leftrightarrow du = -\frac{1}{(y+1)^2} dy$

$= \int_{+\infty}^{1} g(y)\left(-\frac{2}{(y+1)^2}\right) dy = \int_1^{+\infty} g(y) \frac{2}{(y+1)^2} dy$

donc $\frac{V}{\mathcal{U}}$ suit la loi à densité $y \mapsto \frac{2}{(y+1)^2}$

$E[Y] = \int_1^{+\infty} \frac{2y}{(y+1)^2} dy$

$u = (y+1)^2 \Rightarrow \sqrt{u} = y+1 \Rightarrow \sqrt{u} - 1 = y$

$du = (2y+2)\,dy$ $\qquad \frac{dy}{du} = \frac{1}{2\sqrt{u}}$

$\Rightarrow$ $\qquad \Rightarrow dy = \frac{1}{2\sqrt{u}}\,du$

$$\int_4^{+\infty} \frac{\sqrt{u}-1}{u\sqrt{u}}\,du = \int_4^{+\infty} \frac{1}{u}\,du - \int_4^{+\infty} \frac{1}{u\sqrt{u}}\,du$$

$$= [\ln(u)]_4^{+\infty} - \int_4^{+\infty} u^{-\frac{3}{2}}\,du$$

$$= [\ln(u)]_4^{+\infty} - [-2u^{-\frac{1}{2}}]_4^{+\infty}$$

$$= [\ln(u)]_4^{+\infty} + 2\left[\frac{1}{\sqrt{u}}\right]_4^{+\infty} = +\infty$$

Exercice 3

$X \sim Exp(\lambda)$ donc $f_X(x) = \lambda e^{-\lambda x}$

1) $Y = e^{-\lambda X}$

$$\int g(Y)\,dP_Y = \int g(e^{-\lambda X})\,dP_X$$

$$= \int_0^{+\infty} g(e^{-\lambda x})\,\lambda e^{-\lambda x}\,dx$$

$y = e^{-\lambda x}$ $\qquad y_0 = e^{-\lambda \cdot 0} = 1$

$\qquad y_1 = e^{-\lambda \infty} = e^{-\infty} = 0$

$\frac{dy}{dx} = -\lambda e^{-\lambda x}$

$dx = \frac{dy}{-\lambda \underbrace{e^{-\lambda x}}_{y}} = \frac{dy}{-\lambda y}$

$$= -\int_1^0 g(y)\,dy = \int_0^1 g(y)\,dy$$

donc Y suit la loi uniforme car est à densité

$\mathbb{1}_{[0,1]}$

2) $Y = \sqrt{2\lambda X}$

$$\int g(Y)\,dP_Y = \int g(\sqrt{2\lambda X})\,dP_X \qquad (2\lambda x)^{\frac{1}{2}}$$

$$= \int_0^{+\infty} g(\sqrt{2\lambda x})\,\lambda e^{-\lambda x}\,dx \qquad \frac{1}{\sqrt{2\lambda x}}$$

$$y = \sqrt{2\lambda x} \Rightarrow y^2 = 2\lambda x \qquad y_0 = 0$$
$$\Rightarrow x = \frac{y^2}{2\lambda} \qquad y_1 = +\infty$$

$$\frac{dx}{dy} = \frac{y}{\lambda} \Rightarrow dx = \frac{y}{\lambda}\,dy$$

$$\to \int_0^{+\infty} g(y)\,\lambda e^{-\lambda\frac{y^2}{2\lambda}}\,\frac{y}{\lambda}\,dy$$

$$= \int_0^{+\infty} g(y)\,y e^{-\frac{y^2}{2}}\,dy$$

donc Y suit la loi à densité $y \mapsto y e^{-\frac{y^2}{2}}$

3) Notons Φ CDF de la loi normale standarde.

D'après 1) $e^{-\lambda X} \sim Unif([0,1])$

alors $\Phi^{-1}(\underbrace{e^{-\lambda X}}_{U}) \sim N(0,1)$

Par contre, on ne sait pas Φ, alors, on utilise la méthode de Box-Muller.

D'après 1) $e^{-\lambda X} \sim Unif([0,1])$ donc

$$2\pi e^{-\lambda X} \sim Unif([0,2\pi])$$

$\lambda X \sim Exp(1)$

donc $\underbrace{\sqrt{2\lambda X}\,\sin(2\pi e^{-\lambda X})}_{Y} \sim N(0,1)$

$(Z_1, Z_2) \overset{i.i.d}{\sim} N(0,1)$

donc $R = \sqrt{Z_1^2 + Z_2^2}$

$$f_{Z_1,Z_2}(x,y) = \frac{1}{2\pi} e^{-\frac{(x^2+y^2)}{2}} = \frac{1}{2\pi} \underbrace{r e^{-\frac{r^2}{2}}}_{\sqrt{2\lambda X}} = f_{R_1}(r,\theta)$$

si $x = r\cos\theta$
$y = r\sin\theta$

$$Z_1 = \sqrt{2\lambda X} \cos 2\pi e^{-\lambda X}$$

$$Z_2 = \sqrt{2\lambda X} \sin 2\pi e^{-\lambda X}$$

Exercice 4

$U \sim \mathcal{U}nif([0,1])$

1] $Y = -\frac{1}{\lambda}\ln(U)$

$$\int g(Y)\, dP_Y = \int g\left(-\frac{1}{\lambda}\ln(U)\right) P_U$$

$$= \int_0^1 g\left(-\frac{1}{\lambda}\ln(u)\right) \mathbb{1}_{[0,1]}(u)\, du$$

$y_0 = +\infty$
$y_1 = 0$

$y = -\frac{1}{\lambda}\ln(u) \Rightarrow -\lambda y = \ln(u) \Rightarrow u = e^{-\lambda y}$

$\frac{dy}{du} = -\frac{1}{\lambda}\frac{1}{u}$

$\Rightarrow \frac{dy}{du} = -\frac{1}{\lambda}\frac{1}{e^{-\lambda y}} \Rightarrow du = -\lambda e^{-\lambda y} dy = du$

$$= \int_{+\infty}^0 g(y) - \lambda e^{-\lambda y} dy = \underbrace{\int_0^{+\infty} g(y)\, \underbrace{\lambda e^{-\lambda y}}_{h} dy}$$

Y suit la loi exponentielle $\underbrace{\lambda e^{-\lambda y}}$

2] $Y = \sqrt{U}$

$$\int g(Y)\, dP_Y = \int g(\sqrt{U})\, dP_U$$

$$= \int_0^1 g(\sqrt{u})\, du$$

$y = \sqrt{u} \Rightarrow u = y^2 \Rightarrow \frac{du}{dy} = 2y \Rightarrow du = 2y\,dy$

$= \int_0^1 g(y) 2y\, dy$ à densité $2y$

3) $\mathbb{1}_{\{u<p\}} = P(\mathbb{1}_{\{u<p\}} = 1) = P(u<p) = p$

$$P(\mathbb{1}_{\{u<p\}} = 0) = 1-p$$

donc $\mathbb{1}_{\{u<p\}}$ suit la loi de Bernoulli.

Exercice 5

1) Notons $g_k(x) = x^k$

$$E[X^k] = E[g_k(X)] = \int g_k(x)\, f(x)\, dx = \int k^x f(x) dx$$

2) Notons $\mu = E[X]$ $\quad h_k(x) = (x-\mu)^k$

$$E((X - E[X])^k) = E[h_k(X)]$$
$$= \int h_k(X)\, dP_X$$
$$= \int h_k(x) f(x) dx$$
$$= \int (x-\mu)^k f(x) dx$$

3) $U = \mathcal{U}([a,b])$ $\quad f_U(u) = \mathbb{1}_{[a,b]}(u)$

$$E[U^k] = \int u^k \mathbb{1}_{[a,b]}(u) du = \int_a^b u^k du = \left[\frac{u^k}{k+1}\right]_a^b = \frac{b^k}{k+1} - \frac{a^k}{k+1}$$

$$E[U] = \frac{b-a}{2}$$

$$E[(U - \underset{\mu}{\underset{\|}{E[U]}})^k] = \int_a^b (u-\mu)^k du$$

$$= \int_a^b \left(\frac{2u-b+a}{2}\right)^k du$$

$$y_0 = \frac{2a-b+a}{2} = \frac{3a-b}{2}$$
$$y_1 = \frac{b+a}{2}$$

$$y = \frac{2u-b+a}{2} = u - \frac{(b-a)}{2}$$

$$\frac{dy}{du} = 1 \implies dy = du$$

$$= \int_{\frac{3a-b}{2}}^{\frac{b+a}{2}} y^k dy = \left[\frac{y^{k+1}}{k+1}\right]_{\frac{3a-b}{2}}^{\frac{b+a}{2}}$$

$X \sim Exp(\lambda)$ $\quad f_X(x) = \lambda e^{-\lambda x}$

$$E[X^k] = \int_0^{+\infty} x^k \lambda e^{-\lambda x} dx$$

$f = x^k \quad g' = \lambda e^{-\lambda x}$

$f' = k x^{k-1} \quad g = -e^{-\lambda x}$

$$(fg)' = f'g + fg' \Rightarrow f'g = (fg)' - fg'$$

$\lambda \quad \mu$

$$\int g \, d\lambda = \int g \, d\mu \Rightarrow \lambda = \mu$$

$X \sim \dots \quad f \quad$ PDF $\quad f dx$

$Y = f(X)$

$$\int g(Y) \, dP_Y = \underbrace{\int g(f(X)) \, dP_X}$$

$\parallel$

$f(X)$

$$= \int_a^b g(f(x)) f(x) \, dx$$

$\int g(y) h(y) dy \Leftarrow \quad y = f(x)$

Exercice 6

$$g_{a,\lambda}(x) = c\lambda^a x^{a-1} e^{-\lambda x}$$

0) $$c \int_0^{+\infty} x^{a-1} \lambda^a e^{-\lambda x}\, dx$$

$$J_a = \int_0^{+\infty} x^{a-1} \lambda^a e^{-\lambda x} = \int_0^{+\infty} \frac{u^{a-1}}{\lambda^{a-1}} \lambda^{a-1} e^{-u} du$$

$u = \lambda x \Rightarrow x = \frac{u}{\lambda}$ $\quad = \int_0^{+\infty} u^{a-1} e^{-u} du = \Gamma(a)$

$du = \lambda dx$

d'où $\quad c = \frac{1}{\Gamma(a)}$

1) $$g_{1,\lambda}(x) = \frac{1}{\Gamma(1)} \lambda^1 x^0 e^{-\lambda x} = \lambda e^{-\lambda x}$$

ce qui est la fonction de densité de la loi exponentielle.

2) $Z \sim G(a,1) \quad \lambda > 0$

$X = \frac{Z}{\lambda}$

$$E[f(X)] = E[f(\tfrac{Z}{\lambda})] = \int_0^{+\infty} f(\tfrac{x}{\lambda}) \frac{1}{\Gamma(a)} x^{a-1} e^{-x} dx$$

$u = \frac{x}{\lambda} \Rightarrow x = \lambda u$

$du = \frac{1}{\lambda} dx \Rightarrow dx = \lambda du$

$$= \int_0^{+\infty} f(u) \frac{1}{\Gamma(a)} (\lambda u)^{a-1} e^{-\lambda u} \lambda du$$

$$= \int_0^{+\infty} f(u) \frac{1}{\Gamma(a)} \lambda^a u^{a-1} e^{-\lambda u} du$$

ce qui est la loi $G(a,\lambda)$

3) $Z = \lambda X$

$$E[Z] = \frac{1}{\Gamma(a)} \int_0^{+\infty} x^a e^{-x} dx = \frac{\Gamma(a+1)}{\Gamma(a)} = a\,?$$

$$E[Z^2] = \frac{1}{\Gamma(a)} \int_0^{+\infty} x^{a+1} e^{-x} dx = \frac{\Gamma(a+2)}{\Gamma(a)} = (a+1)(a) = a^2 + a$$

$$E[Z^2] - E[Z]^2 = a^2 + a - a^2 = a$$

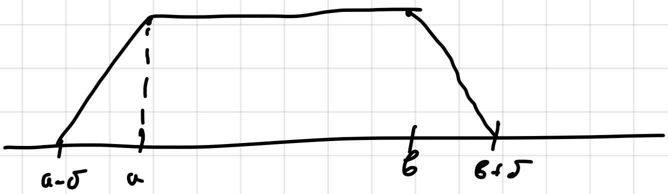
Exercice 7

1) μ, ν $\quad \int_{\mathbb{R}} f(x)\, d\mu(x) = \int_{\mathbb{R}} f(x)\, d\nu(x) \quad \delta > 0$

$[a,b]$

(a)

$$f(x) = \begin{cases} 0 & \text{si } x < a-\delta \\ \frac{1}{\delta}x + 1 - \frac{a}{\delta} & \text{si } x \in [a-\delta, a[\\ 1 & \text{si } x \in [a,b] \\ -\frac{1}{\delta}x + 1 + \frac{b}{\delta} & \text{si } x \in]b, b+\delta] \\ 0 & \text{sinon} \end{cases}$$



$yx + z$

$$\begin{cases} ya + z = 1 \\ y(a-\delta) + z = 0 \end{cases}$$

$ya - y(a-\delta) = 1$

$y(a - a + \delta) = 1$

$y(\delta) = 1 \iff y = \frac{1}{\delta}$

$\frac{1}{\delta}a + z = 1 \iff z = 1 - \frac{1}{\delta}a$

b) $\int_a^b f(x) \quad d\mu(x) \leq \int_{a-\delta}^{b+\delta} f(x)\, d\nu(x)$

d'où $\mu([a,b]) \leq \nu([a-\delta, b+\delta])$

c) $\delta > 0$ était arbitraire, en faisant $\delta \to 0$ on obtient $\mu([a,b]) \leq \nu([a,b])$

par preuve symétrique on obtient l'inégalité réciproque, d'où l'égalité.

2) $\mathcal{A} = \{B \in \mathcal{B}(\mathbb{R}) : \mu(B) = \nu(B)\}$

a) Soient $A, B \in \mathcal{A}$ tq $A \subset B$

$B \setminus A = \{w \in B : w \notin A\} \in \mathcal{B}(\mathbb{R})$

$\mu(B\setminus A) = \mu(B) - \mu(A) = \nu(B) - \nu(A)$ donc $B \setminus A \in \mathcal{A}$

$(A_n)_{n\in\mathbb{N}}$ $\forall n\in\mathbb{N}$ $A_n\in\mathcal{A}$

Notons $B_n = A_n \setminus A_{n-1}$ avec $B_0 = A_0$

D'après a) $B_n \in \mathcal{A}$ $\forall n\in\mathbb{N}$

$\mu(B_n) = \mu(A_n) - \mu(A_{n-1})$ avec tous B_n disjoints

i.e $B_n \cup B_{n+1} = \mu(B_n) + \mu(B_{n+1})$

D'où $\mu\left(\bigcup_{n=1}^{N} A_n\right) = \mu\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \mu(A_0) + \mu(A_1) - \mu(A_0) + \dots$

$+ \mu(A_N) - \mu(A_{N-1}) = \mu(A_N)$

donc $\mu\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \lim_{N\to+\infty} \mu(A_N) = \lim_{N\to+\infty} \nu(A_N)$

donc $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{A}$

c) $\mathbb{R} \in \mathcal{B}(\mathbb{R})$ pour tout intervalle dans $\mathbb{R}$ $\mu = \nu$ donc pour $\mathbb{R}$ aussi. ATTENTION ⚠

donc $\forall A\in\mathcal{A}$ $\mathbb{R}\setminus A \in \mathcal{A}$ donc $\mathcal{A}$ stable par complémentaire et par union dénombrable.

d'où $\mathcal{A}$ est une tribu.

3) Donc $\mathcal{A}$ est une tribu et $(\mathcal{A},\mu) = (\mathcal{A},\nu)$

d'où $\mu = \nu$