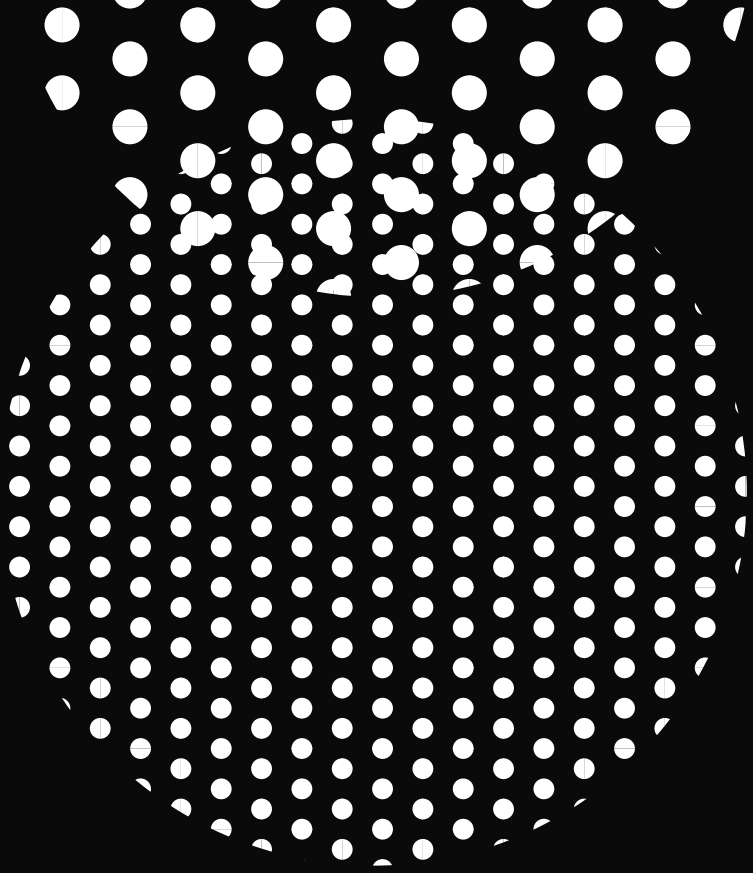

Exercice 4

$\Rightarrow 1$ Supposons que X est réseau déplacé

$$f_X(u) = E[e^{iuX}] = \int_{\mathbb{R}} e^{iux}\, dP_X$$

$$= \sum_{n\in\mathbb{N}} e^{iu(a+nd)} P(X=a+nd)$$

$e^{iua} e^{iund}$

donc $\forall u$ $|f_X(u)| = \left|\sum_{n\in\mathbb{N}} e^{iu(a+nd)} P(X=a+nd)\right|$

$$= \left|\sum_{n\in\mathbb{N}} e^{iu(a+nd)} P(X=a+nd)\right|$$

$$= \left|e^{iua} \sum_{n\in\mathbb{N}} e^{iund} P(X=a+nd)\right|$$

$$= \left|e^{iua} \sum_{n\in\mathbb{N}} e^{i2\pi n} P(X=a+nd)\right| \quad \text{si } u = \frac{2\pi}{d}$$

donc $|f_X(u)| \leq 1$

$$= \left|e^{i\frac{2\pi}{d}a} \sum_{n\in\mathbb{N}} P(X=a+nd)\right|$$

$$= \left|e^{i\frac{2\pi}{d}a}\right| = 1$$

$\Leftarrow 1$ Supposons que $\exists\, u \neq 0$ tq $|f_X(u)| = 1$,
prennons donc tel u.

$e^{iuX} = e^{i\theta}$ i.e $uX = \theta + 2k\pi$ avec $k \in \mathbb{Z}$

$\Leftrightarrow X = \frac{\theta}{u} + \frac{2\pi}{u}k$

donc en posant $a = \frac{\theta}{u}$ et $\frac{2\pi}{d}$ on obtient que
X est réseau déplacé.

X rés.dep $\Rightarrow$ $|f_X|$ périodique

Soit $T = \frac{2\pi}{d}$ donc

$\forall t \in \mathbb{R}$ $f_X(t+T) = E[e^{i(t+T)X}]$

$$= \sum_{n\in\mathbb{N}} e^{it(a+nd)} e^{iT(a+nd)} P(X=a+nd)$$

$$= \sum_{n\in\mathbb{N}} e^{it(a+nd)} e^{i2\pi\frac{a}{d}} \underbrace{e^{i2\pi n}}_{=1} P(X=a+nd)$$

$$= e^{i2\pi\frac{a}{d}} \sum_{n\in\mathbb{N}} e^{it(a+nd)} P(X=a+nd) = e^{i2\pi\frac{a}{d}} f_X(t)$$

donc $|f_X(t+T)| = |e^{i2\pi \frac{a}{d}} f_X(t)| = |f_X(t)|$

donc $|f_X|$ est périodique de période T.

$|f_X|$ périodique $\Rightarrow \exists u \neq 0 \; |f_X(u)| = 1$

Supposons que $|f_X|$ est périodique de période $T > 0$

Or $|f_X(0)| = 1$ donc $|f_X(T)| = 1$ d'où la conclusion.

2. Supposons que X est réseau donc

$$\sum_{n \in \mathbb{Z}} P(X = nd) = 1$$

Soit $u \in \mathbb{R}$

$$f_X(u) = \sum_{n \in \mathbb{Z}} e^{iund} P(X = nd)$$

en posant $u = \frac{2\pi}{d}$, on a

$$f_X\left(\frac{2\pi}{d}\right) = \sum_{n \in \mathbb{Z}} e^{i2\pi n} P(X = nd)$$

$$= \sum_{n \in \mathbb{Z}} P(X = nd) = 1$$ d'où la conclusion.

$\Leftarrow$ Supposons que $\exists u \neq 0$ tq $f_X(u) = 1$

Or $|f_X(u)| = |1| = 1$ donc X est réseau déplacé

i.e $\sum_{n \in \mathbb{Z}} P(X = a + nd) = 1$

de plus $\sum_{n \in \mathbb{Z}} e^{iu(a+nd)} P(X = a + nd) = 1$

i.e $e^{iua} \sum_{n \in \mathbb{Z}} e^{iund} P(X = a + nd) = 1$

posons $u = \frac{2\pi}{d}$

donc $e^{i\frac{2\pi}{d}a} \sum_{n \in \mathbb{Z}} P(X = a + nd) = 1$

donc $e^{i\frac{2\pi}{d}a} = 1$

donc $\frac{2\pi}{d} u = k 2\pi$ avec $k \in \mathbb{Z}$

d'où $\frac{u}{d} = k$ i.e $\frac{a}{d} \in \mathbb{Z}$

donc $u = 0$ convient

3)

Exercice 1

$X \sim Ber(p)$ $\quad P(X=1) = p$

alors pour $u \in \mathbb{R}$ $\quad \varphi_X(u) = E[e^{iuX}] = p\, e^{iu} + (1-p)$

$X \sim Geo(p)$

$$\varphi_X(u) = E[e^{iuX}] = \sum_{k=1} e^{iuk} p(1-p)^{k-1} = e^{iu} \sum_{k=0} e^{iuk} p(1-p)^k$$

$$= e^{iu} p \sum_{k=0} \left(e^{iu}(1-p)\right)^k$$

$$= \frac{e^{iu} p}{1 - e^{iu}(1-p)}$$

$X \sim \mathcal{P}(\lambda)$

$$\varphi_X(u) = E[e^{iuX}] = \sum_{k=0} e^{iuk} e^{-\lambda} \frac{\lambda^k}{k!}$$

$$= \sum_{k=0} e^{iuk-\lambda} \frac{\lambda^k}{k!}$$

$$= e^{-\lambda} \sum \frac{(e^{iu}\lambda)^k}{k}$$

$$= e^{-\lambda} e^{e^{iu}\lambda} = e^{\lambda(e^{iu}-1)}$$

$X \sim Exp(\alpha)$

$$\varphi_X(u) = E[e^{iuX}] = \int_{\mathbb{R}^+} e^{iux} \alpha e^{-\alpha x} dx$$

$$= \alpha \int_{\mathbb{R}^+} e^{iux - \alpha x} dx$$

$$= \alpha \int_{\mathbb{R}^+} e^{x(iu-\alpha)} dx$$

$$= \alpha \left[\frac{e^{x(iu-\alpha)}}{iu-\alpha} \right]_0^{+\infty} = 0 - \frac{\alpha}{iu-\alpha}$$

$$= \frac{\alpha}{\alpha - iu}$$

$X \sim Unif([a,b])$

$$\varphi_X(u) = E[e^{iuX}] = \int_a^b e^{iux} dx = \left[\frac{e^{iux}}{iu} \right]_a^b = \frac{e^{iub} - e^{iua}}{iu}$$

Exercice 2

$$\varphi_X(u) = \frac{\alpha}{\alpha - iu} \qquad \varphi_Y(u) = \frac{\alpha}{\alpha - iu}$$

$$Z = X - Y$$

$$\varphi_Z(u) = E[e^{iu(X-Y)}] = E[e^{iuX}]E[e^{-iuY}]$$

$$= \varphi_X(u)\,\varphi_Y(-u)$$

$$= \frac{\alpha^2}{(\alpha - iu)(\alpha + iu)}$$

$$= \frac{\alpha^2}{\alpha - |iu|^2} = \frac{\alpha^2}{\alpha^2 + u^2} = \varphi_Z(u)$$

$$E[e^{iuZ}] = \iint_{\mathbb{R}^+ \times \mathbb{R}^+} e^{iu(x-y)}\, \alpha e^{-\alpha x}\, \alpha e^{-\alpha(y)}\, dx\, dy$$

$$= \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} e^{iu(x-y)}\, \alpha^2 e^{-\alpha(x+y)}\, dx\, dy$$

$$z = x - y \qquad \Rightarrow x = z + y$$

$$dz = dx \qquad z_+ = +\infty$$
$$z_- = -y$$

$$= \int_{\mathbb{R}^+} \int_{-y}^{+\infty} e^{iuz}\, \alpha^2 e^{-\alpha(z+2y)}\, dz\, dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuz}\, \alpha^2 e^{-\alpha(z+2y)}\, \mathbb{1}_{\{y \geq 0\}}(y)\, \mathbb{1}_{\{z \geq -y\}}(z)\, dz\, dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuz}\, \alpha^2 e^{-\alpha(z+2y)}\, \mathbb{1}_{\{y \geq 0\}}(y)\, \mathbb{1}_{\{z \geq -y\}}(z)\, dz\, dy$$

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} e^{iuz}\, \alpha^2 e^{-\alpha(z+2y)}\, \mathbb{1}_{\{y \geq 0\}}(y)\, \mathbb{1}_{\{z+y \geq 0\}}(z)\, dz\, dy$$

$$= \int_{\mathbb{R}} e^{iuz} e^{-\alpha z} \alpha \left(\int_{\mathbb{R}} \alpha\, e^{-\alpha 2y}\, \mathbb{1}_{\{y \geq 0\}}(y)\, \mathbb{1}_{\{z+y \geq 0\}}(z)\, dy \right) dz$$

$$\int_{\mathbb{R}} \alpha\, e^{-\alpha 2y}\, \mathbb{1}_{\{y \geq 0\}}(y)\, \mathbb{1}_{\{z+y \geq 0\}}(z)\, dy = (*)$$

Si $z \geq 0$ $\quad (*) = \int_0^{+\infty} \alpha e^{-\alpha 2y}\, \mathbb{1}_{\{y \geq 0\}}(y)\, dy$

$$= \left[-\frac{\alpha e^{-\alpha 2y}}{2\alpha} \right]_0^{+\infty} = \frac{1}{2}$$

Si $z < 0$ $\quad (*) = \int_{-z}^{+\infty} \alpha e^{-\alpha 2y} = \frac{1}{2} e^{\alpha 2z}$

densité de $X - Y$ $\quad \dfrac{\alpha e^{-\alpha |z|}}{2}$

3] $W \sim \text{Cauchy}(\lambda)$

$$\varphi_W(u) = E[e^{iuW}] = \int_{\mathbb{R}} e^{iux} \frac{\lambda}{\pi(x^2+\lambda^2)} dx$$

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itx} \varphi_X(t) dt$$

$$\cancel{\frac{\alpha^2}{\alpha^2+a^2} = \varphi_Z(u)}$$

$$\cancel{f(x) = \frac{1}{\lambda} \frac{1}{\pi} \varphi_Z(x)}$$

$$\frac{\alpha e^{-\alpha|z|}}{2} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-itz} \frac{\alpha^2}{\alpha^2+t^2} dt$$

pour $z = -x$

$$\frac{\alpha e^{-\alpha|x|}}{2} = \frac{1}{2\pi} \int_{\mathbb{R}} e^{itx} \frac{\alpha^2}{\alpha^2+t^2} dt$$

$$\frac{2}{\alpha} \frac{\alpha e^{-\alpha|x|}}{2} = \int_{\mathbb{R}} e^{itx} \frac{\alpha}{\pi(\alpha^2+t^2)} dt$$

$= e^{-\alpha|x|}$ est la fonction caractéristique de la loi de Cauchy

4.

$$\varphi_{\alpha U+\beta V}(u) = \varphi_{\alpha U}(u)\, \varphi_{\beta V}(u) = \varphi_U(\alpha u)\, \varphi_V(\beta u)$$

$$= e^{-\lambda|\alpha u|} e^{-\mu|\beta u|}$$

$$= e^{-\lambda\alpha|u|} e^{-\mu|\beta u|}$$

$$= e^{-(\lambda\alpha+\mu\beta)|u|}$$

donc $\alpha U+\beta V \sim \text{Cauchy}(\alpha\lambda+\beta\mu)$

Exercice 3

1. Soit $u \in \mathbb{R}$ $\quad \varphi_{X+Y}(u) = E[e^{iu(X+Y)}]$

$$= E[e^{iuX+iuY}]$$

$$= E[e^{iuX}]\, E[e^{iuY}]$$

$$= \varphi_X(u)\, \varphi_Y(u)$$

2. Par récurrence d'après 1. $\varphi_{X_1+\dots+X_n}(u) = \varphi_{X_1}(u) \dots \varphi_{X_n}(u)$

3. Notons $Y_1 \sim \mathcal{B}(p)$ donc $\varphi_{Y_1}(u) = 1-p+pe^{iu}$

On note $X_n = Y_1+\dots+Y_n \sim \mathcal{B}(n,p)$

$$\varphi_{X_n}(u) = (1-p+pe^{iu})^n = \left(1+p(e^{iu}-1)\right)^n$$

4. $\varphi_{X_k}(u) = e^{\lambda_k(e^{iu}-1)}$

$$\varphi_{X_1+\dots+X_n} = e^{(e^{iu}-1)(\lambda_1+\dots+\lambda_n)}$$

5. $\varphi_{X_k}(u) = \dfrac{\lambda_k}{\lambda_k - iu}$

$$\varphi_{X_1+\dots+X_n}(u) = \frac{\lambda_1 \cdot \ldots \cdot \lambda_n}{(\lambda_1 - iu)\cdot \ldots \cdot (\lambda_n - iu)}$$

6. $\varphi_{X_k}(u) = e^{ium_k - \sigma_k^2 \frac{u^2}{2}}$

$$\varphi_{X_1+\dots+X_n}(u) = e^{iu(m_1+\dots+m_n) - \frac{1}{2}u^2(\sigma_1^2+\dots+\sigma_n^2)}$$