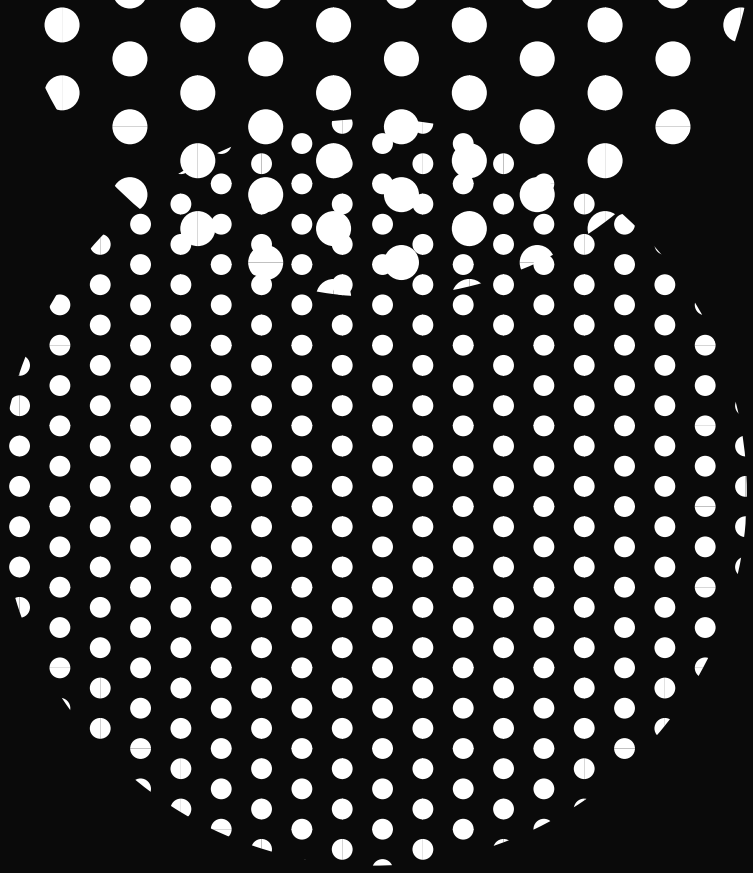

Exercice 4

$$E[f(X)] = \int_{\Omega} f(x)\, d\mathbb{P}(x)$$
$$\int_0^1 = f(2w)\, dw$$

1) $Y_a \sim G(a)$ $\quad Y_b \sim G(b)$

$$U = \frac{Y_a}{(Y_a + Y_b)}$$

$$E[f(U)] = \left[f\left(\frac{Y_a}{Y_a+Y_b}\right)\right] = \int_0^{+\infty}\int_0^{+\infty} f\left(\frac{x}{x+y}\right) f_{Y_a,Y_b}(x,y)\, dx\, dy$$

$$= \int_0^{+\infty}\int_0^{+\infty} f\left(\frac{x}{x+y}\right) f_{Y_a}(x)\, f_{Y_b}(y)\, dx\, dy \quad \text{car } Y_a, Y_b \text{ indépts.}$$

$$u = \frac{x}{x+y} \Rightarrow y = \frac{x}{u} - x \Rightarrow dy = -\frac{x}{u^2}\, du$$

$$\hookrightarrow \int_0^{+\infty}\left(\int_0^1 f(u)\, f_{Y_a}(x)\, f_{Y_b}\left(\frac{x}{u} - x\right) \frac{x}{u^2}\, du\right) dx$$

$$= \int_0^{+\infty}\int_0^1 f(u)\, \Gamma(a)^{-1} x^{a-1} e^{-x}\, \Gamma(b)^{-1}\left(\frac{x}{u} - x\right)^{b-1} e^{x - \frac{x}{u}}\, \frac{x}{u^2}\, du\, dx$$

$$= \int_0^{+\infty} \Gamma(a)^{-1}\Gamma(b)^{-1} \int_0^1 f(u)\, x^a \left(\frac{x}{u} - x\right)^{b-1} e^{-\frac{x}{u}} \frac{1}{u^2}\, du\, dx$$

$$= \int_0^1 \Gamma(a)^{-1}\Gamma(b)^{-1} f(u) \left(\int_0^{+\infty} \frac{1}{u^2} x^a \left(\frac{x}{u} - x\right)^{b-1} e^{-\frac{x}{u}}\, dx\right) du$$

$$= \int_0^1 \Gamma(a)^{-1}\Gamma(b)^{-1} f(u) \left(\int_0^{+\infty} \frac{1}{u^2} x^a x^{b-1} \left(\frac{1}{u} - 1\right)^{b-1} e^{-\frac{1}{u}x}\, dx\right) du$$

$$= \int_0^1 \Gamma(a)^{-1}\Gamma(b)^{-1} f(u) \frac{1}{u^2}\left(\frac{1}{u} - 1\right)^{b-1} \left(\int_0^{+\infty} x^{a+b-1} e^{-\frac{1}{u}x}\, dx\right) du$$

$s = \frac{1}{u}x$

$\Rightarrow x = us$

$dx = u\, ds$

$$= \int_0^1 \Gamma(a)^{-1}\Gamma(b)^{-1} f(u) \frac{1}{u^2}\left(\frac{1}{u} - 1\right)^{b-1} \left(\int_0^{+\infty} u s^{a+b-1} e^{-s} u\, ds\right) du$$

$$= \int_0^1 \Gamma(a)^{-1}\Gamma(b)^{-1} f(u) \frac{1}{u}\left(\frac{1}{u} - 1\right)^{b-1} u^{a+b-1} \left(\int_0^{+\infty} s^{a+b-1} e^{-s}\, ds\right) du$$

$$= \int_0^1 f(u) \frac{1}{\Gamma(a)\Gamma(b)} u^{a-1} (1-u)^{b-1} \Gamma(a+b)\, ds$$

$$= \int_0^1 f(u) \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1} (1-u)^{b-1}\, ds$$

Donc U suit la loi à densité

$$u \longmapsto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} u^{a-1} (1-u)^{b-1} \mathbb{1}_{]0,1[}(u)$$

donc la loi $\mathrm{Beta}(a,b)$

2) $E[Y_a + Y_b] = E[Y_a] + E[Y_b] = a + b$

$E[Y_a] = a$

$$\int_0^1 x^a (1-x)^{b-1}\, dx = B(a+1, b)$$

$$E[U] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+1)\Gamma(b)}{\Gamma(a+1+b)} = \frac{a}{a+b}$$

$$E[U^2] = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(a+2)\Gamma(b)}{\Gamma(a+2+b)} = \frac{(a+1)a}{(a+1+b)(a+b)}$$

$$Var(U) = E[U^2] - E[U]^2 = \frac{(\alpha+1)\alpha}{(\alpha+1+\beta)(\alpha+\beta)} - \frac{\alpha^2}{(\alpha+\beta)^2}$$

$$= \frac{(\alpha+1)\alpha(\alpha+\beta) - (\alpha+1+\beta)\alpha^2}{(\alpha+1+\beta)(\alpha+\beta)^2} = \frac{\alpha((\alpha+1)(\alpha+\beta) - (\alpha+1+\beta)\alpha)}{(\alpha+1+\beta)(\alpha+\beta)^2}$$

$$= \frac{\alpha(\alpha^2+\alpha+\alpha\beta+\beta - \alpha^2 - \alpha - \alpha\beta)}{(\alpha+1+\beta)(\alpha+\beta)^2} = \frac{\alpha\beta}{(\alpha+\beta+1)(\alpha+\beta)^2}$$

<u>Exercices</u> $\quad x \mapsto \frac{\Gamma(\alpha+1-\alpha)}{\Gamma(\alpha)\Gamma(1-\alpha)} x^{\alpha-1}(1-x)^{1-\alpha} \qquad Z: x \mapsto e^{-x}$

$U \sim \beta(\alpha, 1-\alpha) \quad \alpha \in]0,1[\qquad Z \sim Exp(1)$ indépts

$X = UZ$

$$E[f(UZ)] = \int_0^1\int_0^{+\infty} f(uz)\, f_{U,Z}(u,z)\, du\, dz$$

$$\| $$

$$E[f(X)]$$

$$= \int_0^1\int_0^{+\infty} f(uz)\, f_U(u)\, f_Z(z)\, du\, dz \quad \text{car } U \text{ et } Z \text{ indépendantes.}$$

$x = uz \qquad z = \frac{x}{u}$

$dx = u\,dz \Rightarrow dz = \frac{dx}{u}$

$$= \int_0^1\int_0^{+\infty} f(x)\, f_U(u)\, f_Z\left(\frac{x}{u}\right)\frac{1}{u}\, dx\, du$$

$$= \int_0^1\int_0^{+\infty} f(x)\, \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\, u^{\alpha-2}(1-u)^{-\alpha}\, e^{-\frac{x}{u}}\, dx\, du$$

$$= \int_0^{+\infty} f(x)\, \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\int_0^1 u^{\alpha-2}(1-u)^{-\alpha}\, e^{-\frac{x}{u}}\, du\right) dx$$

$y = \frac{x}{u} \qquad dy = -\frac{x}{u^2}\, du$

$\Rightarrow u = \frac{x}{y} \qquad \Rightarrow du = -\frac{u^2}{x}\, dy$

$= -\frac{x}{y^2}\, dy$

$$= \int_0^{+\infty} f(x)\, \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\int_x^{+\infty} \left(\frac{x}{y}\right)^{\alpha-2} \left(1-\frac{x}{y}\right)^{-\alpha} e^{-y}\, \frac{x}{y^2}\, dy\right) dx$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\int_x^{+\infty} \left(\frac{x}{y}\right)^{\alpha-1} \frac{1}{y}\, (y-x)^{-\alpha} \left(\frac{1}{y}\right)^{-\alpha} e^{-y}\, dy\right) dx$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \left(\int_x^{+\infty} x^{\alpha-1} \left(\frac{1}{y}\right)^{\alpha} \left(\frac{1}{y}\right)^{-\alpha} (y-x)^{-\alpha}\, e^{-y}\, dy\right) dx$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)}\, x^{\alpha-1} \left(\int_x^{+\infty} (y-x)^{-\alpha}\, e^{-y}\, dy\right) dx$$

$x^{\alpha-1} e^{-x}$

$$= \int_0^1 \int_0^{+\infty} f(uz)\, f_U(u)\, f_Z(z)\, du\, dz$$

$$= \int_0^1 \int_0^{+\infty} f(uz) \frac{1}{\Gamma(a)\Gamma(1-a)} u^{a-1} (1-u)^{-a} e^{-z}\, du\, dz$$

$$x = uz \qquad u = \frac{x}{z}$$

$$dx = z\, du \Rightarrow du = \frac{dx}{z}$$

$$\Rightarrow = \int_0^{+\infty} \left(\int_0^z f(x) \frac{1}{\Gamma(a)\Gamma(1-a)} \left(\frac{x}{z}\right)^{a-1} \left(1 - \frac{x}{z}\right)^{-a} e^{-z} \frac{1}{z} dx \right) dz$$

$$\frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^{+\infty} f(x) \left(\int_x^{+\infty} x^{a-1} z^{-a} \left(\frac{z-x}{z}\right)^{-a} e^{-z}\, dz \right) dx$$

$$= \frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^{+\infty} f(x) \left(\int_x^{+\infty} x^{a-1} z^{-a} z^{a} (z-x)^{-a} e^{-z}\, dz \right) dx$$

$$= \frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^{+\infty} f(x)\, x^{a-1} \left(\int_x^{+\infty} (z-x)^{-a} e^{-z}\, dz \right) dx$$

$$s = z - x \qquad ds = dz$$

$$\Rightarrow z = s + x$$

$$= \frac{1}{\Gamma(a)\Gamma(1-a)} \int_0^{+\infty} f(x)\, x^{a-1} \left(\int_0^{+\infty} s^{-a} e^{-x} e^{-s}\, ds \right) dx$$

$$= \frac{1}{\Gamma(a)} \int_0^{+\infty} f(x)\, x^{a-1} e^{-x}\, dx$$

Exercice 6

0) La loi de Z est la loi à densité jointe $f_{X,Y}(x,y)$

1) La densité doit être toujours positive et intégrable à 1

$$\int_{-\infty}^{+\infty} \left(\int_{|y|}^{+\infty} \frac{y}{\pi \sigma^2} e^{-\frac{(x^2+y^2)}{\sigma^2}} dx \right) dy$$

$$= \int_{-\infty}^{+\infty} \frac{y}{\pi} e^{-\frac{y^2}{\sigma^2}} \left(\int_{|y|}^{+\infty} \frac{1}{\sigma^2} e^{-\frac{x^2}{\sigma^2}} dx \right) dy$$

$$x = r\cos(\theta) \qquad J = \begin{pmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{pmatrix}$$

$$y = r\sin(\theta)$$

$$J = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}$$

$$|J| = r\cos(\theta)^2 + r\sin(\theta)^2 = r$$

D'où $$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{+\infty} \frac{4}{\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}} r\,dr\,d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi\sigma^2}\left(\int_0^{+\infty} r e^{-\frac{r^2}{\sigma^2}} dr\right) d\theta$$

$$r\cos(\theta) \geq |r\sin(\theta)| \iff \cos(\theta) \geq |\sin(\theta)|$$
$$\iff |\theta| \leq \frac{\pi}{4}$$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi\sigma^2}\left(\left[-\frac{e^{-\frac{r^2}{\sigma^2}}}{2}\sigma^2\right]_0^{+\infty}\right) d\theta$$
$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{4}{\pi\sigma^2} \frac{\sigma^2}{2}\, d\theta = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{2}{\pi}\, d\theta = \frac{\pi}{2}\,\frac{2}{\pi} = 1$$

Donc f est bien une densité.

2)

$$f_X(x) = \int_{-\infty}^{+\infty} f(x,y)\,dy$$
$$= e^{-\frac{x^2}{\sigma^2}} \int_{-\infty}^{+\infty} \frac{4}{\pi\sigma^2} e^{-\frac{y^2}{\sigma^2}} dy$$
$$= \frac{4}{\pi\sigma^2} e^{-\frac{x^2}{\sigma^2}} \sqrt{\sigma^2\pi}$$

$$f_Y(y) = \frac{4}{\pi\sigma^2} e^{-\frac{y^2}{\sigma^2}} \sqrt{\sigma^2\pi}$$

$$f_X(x) f_Y(y) = 16\, e^{-\frac{x^2-y^2}{\sigma^2}} \neq f_{X,Y}(x,y)$$

Donc X et Y ne sont pas indépendantes.

3)

$u = X - Y$
$v = X + Y$

$u = X - V + X$
$\Rightarrow X = \frac{u+v}{2}$
$\Rightarrow Y = \frac{v-u}{2}$

$$J = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix}$$
$$|J| = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$f_{U,V}(u,v) = \frac{1}{2} f_{X,Y}\left(\frac{u+v}{2}, \frac{v-u}{2}\right) = \frac{4}{\pi\sigma^2}\,\frac{1}{2}\, \exp\left(-\frac{\left(\frac{u+v}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2}{\sigma^2}\right)$$
$$= \frac{2}{\pi\sigma^2} \exp\left(\frac{-(u^2 + \cancel{2uv} + v^2 + v^2 - \cancel{2uv} + u^2)}{4\sigma^2}\right)$$
$$= \frac{2}{\pi\sigma^2} \exp\left(-\frac{u^2+v^2}{\sigma^2}\right) \mathbb{1}_{\{u\geq 0, v\geq 0\}}$$

et $x \geq |y| \iff \frac{u+v}{2} \geq \left|\frac{v-u}{2}\right| \iff u+v \geq |v-u| = |u-v|$
$u+v \geq |u-v| \Rightarrow v \geq 0$
$u \geq 0$

On remarque que

$$f_{U,V}(u,v) = \left(\frac{\sqrt{2}}{\sqrt{\pi}\,\sigma} e^{-\frac{u^2}{\sigma^2}}\right)\left(\frac{\sqrt{2}}{\sqrt{\pi}\,\sigma} e^{-\frac{v^2}{\sigma^2}}\right)$$

ce qui donne l'independance de U et V.

Exercice 7

2) X et Y sont independantes si $f_{X,Y}(x,y) = f_X(x) f_Y(y)$ $\forall (x,y) \in \mathbb{R}^2$

3)