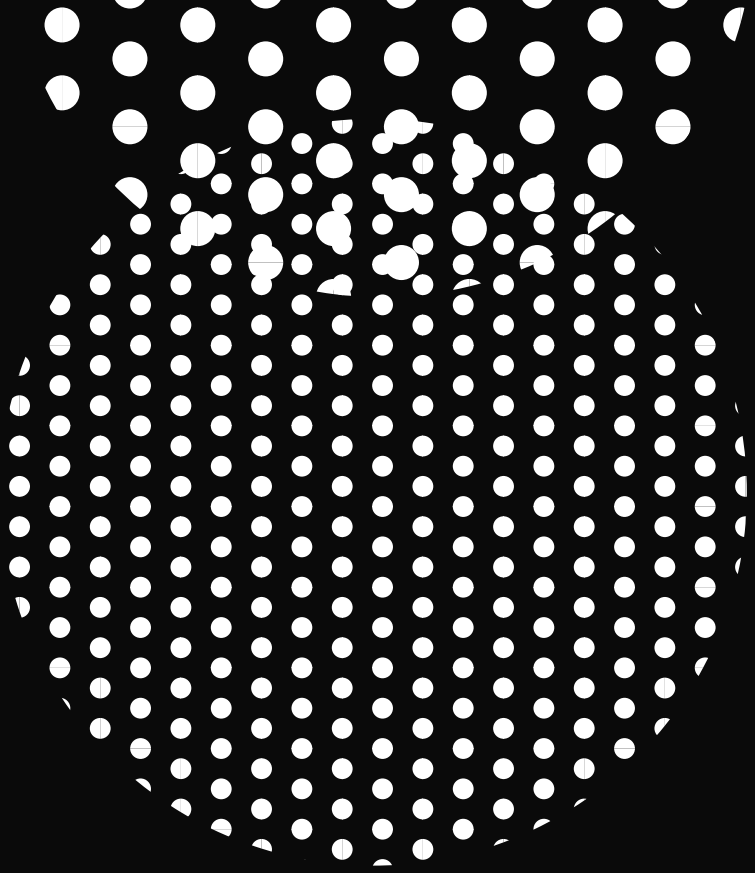

Exercice 3.1

1) $\frac{\partial f}{\partial x}(x,y) = 2x + y$

$\frac{\partial f}{\partial y}(x,y) = 2y + x$

$\nabla f(x,y) = \begin{pmatrix} 2x+y \\ 2y+x \end{pmatrix}$

$\nabla f(x,y) = 0 \iff \begin{cases} 2x+y = 0 \\ 2y+x = 0 \end{cases} \Rightarrow \begin{matrix} y = -2x \\ -4x + x = 0 \end{matrix} \Rightarrow \begin{matrix} y = 0 \\ x = 0 \end{matrix}$

$H_f(0,0) \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad$ $\det H_f(0,0) = 4 - 1 = 3$
$\quad$ $\operatorname{Tr} H_f(0,0) = 4$

Donc $(0,0)$ est un minimum local

4. $g(x,y) = x^2 + y^2 + 4xy - 2$

$\frac{\partial g}{\partial x}(x,y) = 2x + 4y$

$\frac{\partial g}{\partial y}(x,y) = 2y + 4x$

$\nabla f(x,y) = \begin{pmatrix} 2x+4y \\ 2y+4x \end{pmatrix}$ $\quad$ $\nabla f(x,y) = 0 \iff \begin{cases} 2x+4y = 0 \\ 2y+4x = 0 \end{cases}$

$\iff 2y - 8y = 0 \Rightarrow y = 0$
$\Rightarrow x = 0$

Donc $(0,0)$ est un point critique de f

5) $(x+2y)^2 = x^2 + 4xy + 4y^2$

$g(x,y) = (x+2y)^2 - 3y^2 - 2 = [x + 2y + \sqrt{3}y][x + 2y - \sqrt{3}y] - 2$

$\mathcal{L}_1 = \{(x,y) \in \mathbb{R}^2 \mid x = -2y\}$
$\mathcal{L}_2 = \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$

$g(-2y,y) = (-2y+2y)^2 - 3y^2 - 2$

$= -3y^2 - 2 \xrightarrow[\|y\| \to +\infty]{} -\infty$

$g(0,y) = 4y^2 - 3y^2 - 2 = y^2 - 2 \xrightarrow[\|y\| \to +\infty]{} +\infty$ $\quad$ point col

Exercice 3.2

1. $f(x,y) = 2x^3 + 6xy - 3y^2 + 2$

$$\nabla f(x,y) = \begin{pmatrix} 6x + 6y \\ 6x - 6y \end{pmatrix}$$

$$\nabla f(x,y) = 0 \iff \begin{cases} 6x + 6y = 0 \\ 6x - 6y = 0 \end{cases} \Rightarrow x = 0 \text{ et } y = 0$$

$$H_f(x,y) = \begin{pmatrix} 6 & 6 \\ 6 & -6 \end{pmatrix}$$

$\det H_f(x,y) = -36 - 36 < 0$ donc un point selle

2. $g(x,y) = y(x^2 + \ln(y)^2)$ pour $(x,y) \in \mathbb{R} \times \mathbb{R}_+^*$
$= x^2 y + \ln(y)^2 y$

$$\nabla g(x,y) = \begin{pmatrix} 2y \cdot x \\ x^2 + 2\ln(y) + \ln(y)^2 \end{pmatrix}$$

$$\nabla g(x,y) = 0 \iff \begin{cases} 2y \cdot x = 0 \Rightarrow x = 0 \\ x^2 + 2\ln(y) + \ln(y)^2 = 0 \\ 2\ln(y) + \ln(y)^2 = 0 \end{cases}$$

$(0,1)$, $(0, e^{-2})$

$$H_f(x,y) = \begin{pmatrix} 2y & 2x \\ 2x & \frac{2}{y} + 2\ln(y)\frac{1}{y} \end{pmatrix}$$

$$H_f(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$-2\ln(y) = \ln(y)^2$
$-2 = \ln(y)$
$e^{-2} = y$
$1 = y$

3. $h(x,y) = x^4 + y^4 - 4xy$

$$\nabla h(x,y) = \begin{pmatrix} 4x^3 - 4y \\ 4y^3 - 4x \end{pmatrix}$$

$$\nabla h(x,y) = 0 \iff \begin{cases} 4x^3 - 4y = 0 \\ 4y^3 - 4x = 0 \end{cases} \iff \begin{cases} x^3 - y = 0 \\ y^3 - x = 0 \end{cases} \iff \begin{cases} x^3 = y \\ y^3 = x \end{cases}$$

$$\iff \begin{cases} x^9 = x \iff x(x^8 - 1) = 0 \end{cases}$$

$x = 0, 1, -1$
$y = 0, 1, -1$

$$H_f(x,y) = \begin{pmatrix} 12x & -4 \\ -4 & 12y \end{pmatrix}$$

$\det(H_f(0,0)) = 0 - 16 < 0 \Rightarrow$ point selle

$\det(H_f(1,1)) > 0$ et $Tr(H_f(1,1)) > 0$ donc min locale

$\det(H_f(-1,-1)) > 0$ et $Tr(H_f(-1,-1)) < 0$ donc max locale

TD3

Exercice 1-1

1] $f(x,y) = xy$

$\mathcal{L}_1(x,y,p_1) = xy - p_1\left(\frac{1}{x} + \frac{1}{y} - \frac{2}{3}\right)$

$\begin{cases} \frac{\partial \mathcal{L}_1}{\partial x} = 0 \\ \frac{\partial \mathcal{L}_1}{\partial y} = 0 \\ \frac{\partial \mathcal{L}_1}{\partial p} = 0 \end{cases} \Leftrightarrow \begin{cases} y + \frac{p_1}{x^2} = 0 \\ x + \frac{p_1}{y^2} = 0 \\ \frac{1}{x} + \frac{1}{y} = \frac{2}{3} \end{cases} \Rightarrow \begin{matrix} p_1 = -yx^2 \\ p_1 = -xy^2 \end{matrix} \Rightarrow yx^2 = xy^2$

$\Downarrow$

$xy[y-x] = 0$

$x \neq 0$ ou $y \neq 0$ ou $y = x$

$y = x \quad \frac{1}{x} + \frac{1}{x} = \frac{2}{3} = \frac{2}{x} = \frac{2}{3} \Rightarrow x = 3 = y$

Exercice 1-6

1. fermé car préimage d'un fermé $]-\infty, 4]$ par cette fonction continue. Donc $\mathcal{C}$ fermé.

Soit $(x,y) \in \mathcal{C}$

$(x^2-1)^2 + y^2 \leq 4$ donc $\inf (x^2-1)^2 = 0$

donc $y^2 \leq 4 \Rightarrow |y| \leq 2$

inf le $y^2 = 0$ donc $(x^2-1)^2 \leq 4 \Rightarrow$

$x^2 - 1 \leq 2$

$x^2 \leq 3$

$|x| \leq \sqrt{3}$

Donc $\mathcal{C}$ est bornée.

2. f est continue sur un compact donc d'après le cours, f admet un minimum et maximum qui sont atteint.

3)

a) $\vec{\nabla} f(x,y) = \begin{pmatrix} y \\ x \end{pmatrix}$

b) $\mathcal{C}^\circ = \{(x,y) \in \mathbb{R}^2 \mid (x^2-1)^2 + y^2 < 4\}$

$\vec{\nabla} f(x,y) = 0 \Rightarrow (x,y) = (0,0) \in \mathcal{C}^\circ$

est un unique point critique de f sur $\mathcal{C}^\circ$

et $f(0,0) = 0$

4) a) $\partial\mathcal{C} = \bar{\mathcal{C}} \setminus \mathcal{C}^\circ = \{(x,y) \in \mathbb{R}^2 \mid (x^2-1)^2 + y^2 = 4\}$

β) $\mathcal{L} = xy - \lambda((x^2-1)^2 + y^2 - 4)$

$= xy - \lambda x^4 + 2\lambda x^2 - \lambda - \lambda y^2 + 4\lambda$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x} = y - \lambda(2(x^2-1)2x) = 0 \\ \frac{\partial \mathcal{L}}{\partial y} = x - \lambda 2y = x - 2\lambda y = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda} = (x^2-1)^2 + y^2 - 4 = 0 \end{cases}$$

$y = 4\lambda x(x^2-1)$

$x = 2\lambda y$

Comme $x \neq 0$ et $y \neq 0$ sur $(0,0) \notin \partial\mathcal{C}$

On trouve $\lambda = \frac{x}{2y}$

$$\begin{cases} y = 4\frac{x}{2y} x(x^2-1) \\ (x^2-1)^2 + y^2 = 4 \end{cases} \iff \begin{cases} \frac{y^2}{2x^2} = x^2 - 1 \\ (x^2-1)^2 + y^2 = 4 \end{cases}$$

5) Trouver (x,y) points critiques de $\mathcal{L}$

$$\left(\frac{y^2}{2x^2}\right)^2 + y^2 = 4$$

$$\frac{y^4}{4x^4} + y^2 = 4$$

$$y^4 + 4x^4y^2 = 16x^4$$

$$(y^2)^2 + 4x^4(y^2) - 16x^4 = 0$$

$$y^2 = \frac{\sqrt{16x^8 + 4 \cdot 16x^4} - 4x^4}{2}$$

On obtient

$$y^2 = \frac{\sqrt{16x^4} \cdot \sqrt{x^4+4} - 4x^4}{2}$$

$$y^2 = (x^2-1)\,2x^2$$

$$(x^2-1)^2 + (x^2-1)2x^2 = 4$$

$$x^4 - 2x^2 + 1 + 2x^4 - 2x^2 - 4 = 0$$

$$= 3x^4 - 4x^2 - 3 = 0$$

$$= 3(x^2)^2 - 4(x^2) - 3 = 0$$

$$x^2 = \frac{4 \pm \sqrt{16 + 4\cdot 3\cdot 3}}{6} = \frac{+\sqrt{52} + 4}{6} \quad \text{car} \quad x^2 \geq 0$$

$$x_+ = \sqrt{\frac{\sqrt{52}+4}{6}}$$

$$x_- = -\sqrt{\frac{\sqrt{52}+4}{6}}$$

$$\left(x^2-1\right)^2 = -\left(\frac{\sqrt{52}-2}{6}\right)^2 + 4 = y^2 \qquad y_+ = +\sqrt{4 - \left(\frac{\sqrt{52}-2}{6}\right)^2}$$

$$y_- = -\sqrt{4 - \left(\frac{\sqrt{52}-2}{6}\right)^2}$$

On vérifie que (x_+, y_+) est critique.

d'une part $(x_+^2-1)^2 + y_+^2 = 4$

d'autre part x_+^2 solution de $(E) \Leftrightarrow (x_+^2-1)^2 + (x_+^2-1)2x_+^2 = 4$

$$\Rightarrow y_+^2 = (x_+^2-1)\,2x_+^2$$

5 min de f :

$$f(x_-, y_+) = f(x_+, y_-) < 0 = f(0,0)$$

$$f(x_+, y_+) = f(x_-, y_-) > 0 \quad \text{max de } f$$

Min de f est atteint en (x_+, y_-) (x_-, y_+)

Max de f est atteint en (x_+, y_+) et (x_-, y_-)