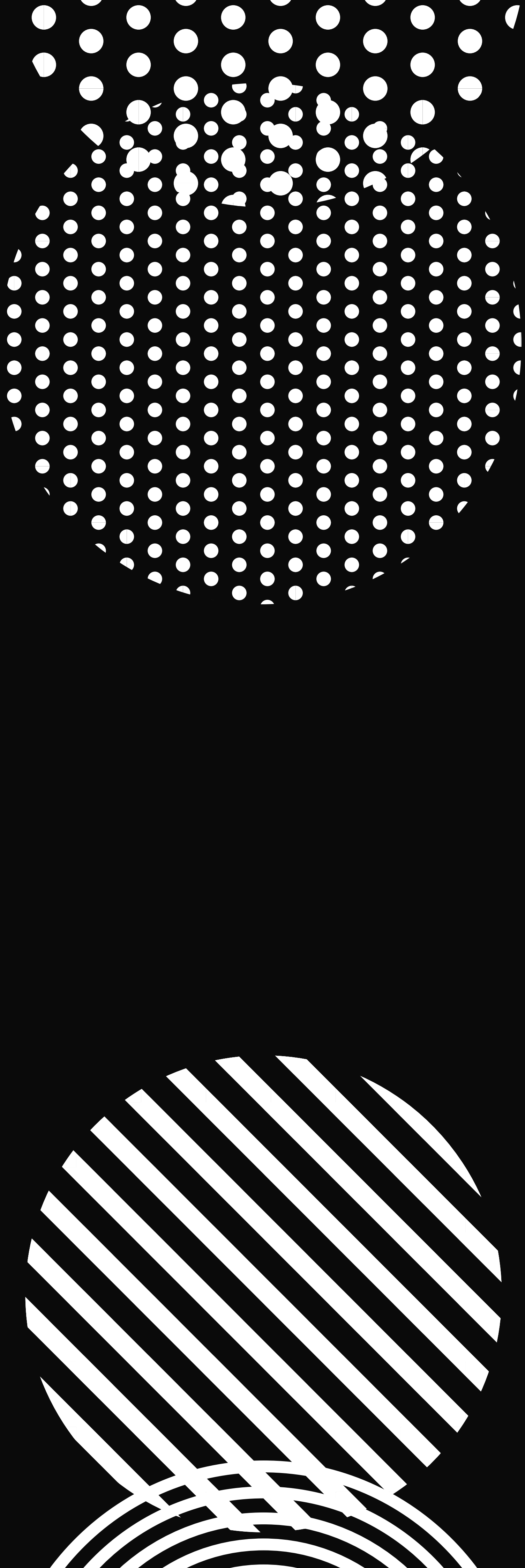

Exercice 1.1

Déf: f est de classe C^1 si f admet des dérivée partielles sur tout point et ces dérivées partielles sont continues.

$\boxed{\Rightarrow}$ Supposons que toutes les dérivées partielles de f sont continues.

$$J_f(x) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$$

Soit $x \in U$

Soit $\varepsilon > 0$ donc $\exists\ \delta_{1,1}, \dots, \delta_{n,m} > 0$

tq $\forall y \in B(x, \delta_{i,j})$

$$\left\| \frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(y) \right\| < \varepsilon$$
$$= \max_{i,j} \left| \frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(y) \right|$$

posons $\delta = \max\{\delta_{1,1}, \dots, \delta_{n,n}\}$

Donc soit $y \in B(x, \delta)$

donc $\| J_f(x) - J_f(y) \|_\infty = \max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| \leq \varepsilon$

d'où la continuité de $x \longmapsto J_f(x)$

$\boxed{\Leftarrow}$ Supposons que $x \mapsto J_f(x)$ continue.

Donc $\forall x \in U$, $\forall \varepsilon > 0$, $\exists\ \delta > 0$ tq $\forall y \in B(x, \delta)$

$$\| J_f(x) - J_f(y) \| < \varepsilon$$
$$= \max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right|$$

Soit $x \in U$, Soit $\varepsilon > 0$

soit $y \in B(x, \delta)$

prenons $\delta > 0$ tq

$$\| J_f(x) - J_f(y) \| = \max_{i,j} \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| < \varepsilon$$

donc $\forall j$ $\left\| \frac{\partial f}{\partial x_j}(x) - \frac{\partial f}{\partial x_j}(y) \right\| = \max_i \left| \frac{\partial f_i}{\partial x_j}(x) - \frac{\partial f_i}{\partial x_j}(y) \right| < \varepsilon$

d'où la continuité des dérivée partielles, donc C^1 de f.

Exercice 1.2

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ et $f(\lambda x + \mu y) = \lambda f(x) + \mu f(y)$

1. Soit $x \in \mathbb{R}^n$ $\quad \frac{\partial f}{\partial x}(x) = g'(t) =$

$$g(t) = f(x + tu) = f(x) + t f(u)$$

$$g'(t) = \frac{f(x+tu) - f(x)}{t} = \frac{f(x) + t f(u) - f(x)}{t} = f(u)$$

$$\frac{\partial f}{\partial x_i}(x) = f(e_i)$$

donc f admet bien les dérivées partielles

et $\forall i \in [\![1, n]\!]$ $\frac{\partial f}{\partial x_i}(x) = f(e_i)$

2) $\underline{Df_x(h) = \sum_i h_i \frac{\partial f}{\partial x_i} = \sum_i h_i f(e_i) = f(h) \text{ car linéaire}}$

$f(x+h) - f(x) = Df_x(h) + o(\|h\|)$

$= f(x) + f(h) + f(x)$

$= f(h)$ continue car f continue.

donc $Df_x(h) = f(h)$

3) Donc $f(x+h) - f(x) = Df_x(h) + o(\|h\|)$

$\Rightarrow f(x+h) = f(x) + Df_x(h) + \cancel{o(\|h\|)}$

pour h fixé $x \mapsto Df_x(h)$ est constante donc continue.
en x alors f est de classe $\mathcal{C}^1$.

Exercice 1.3

$B : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$

donc $\quad B(\lambda x + \mu y, z) = \lambda B(x, z) + \mu B(y, z)$

1. $B(x + h_x, y + h_y) - B(x, y)$

$= B(x, y + h_y) + B(h_x, y + h_y) - B(x, y)$

$= B(x, y) + B(x, h_y) + B(h_x, y) + B(h_x, h_y) - B(x, y)$

$= \underbrace{B(x, h_y) + B(h_x, y)}$ $+ B(h_x, h_y)$

linéaire en h_x et h_y donc

2) $\mathcal{D}_{B_{x,y}}(h_x, h_y) = B(x, h_y) + B(h_x, y)$

$|B(h_x, h_y)| = |\langle h_x, A h_y \rangle| \leq \|h_x\| \underbrace{\|A h_y\|}_{= o(1)} = o(\|h_x\|, \|h_y\|)$

3) polynomiale donc $\mathcal{C}^0$.

Exercice 1.4

$F(x+h) - F(x) = \langle f(x+h), x+h \rangle - \langle f(x), x \rangle$

$= \langle f(x+h), x \rangle + \langle f(x+h), h \rangle - \langle f(x), x \rangle$

$= \langle f(x) + \mathcal{D}f_x(h) + o(\|h\|), x \rangle + \langle f(x) + \mathcal{D}f_x(h) + o(\|h\|), h \rangle$

$- \langle f(x), x \rangle$

$= \langle \mathcal{D}f_x(h) + o(\|h\|), x \rangle + \langle f(x) + \mathcal{D}f_x(h) + o(\|h\|), h \rangle$

$= \langle \mathcal{D}f_x(h), x \rangle + \langle f(x), h \rangle + \langle \mathcal{D}f_x(h), h \rangle + \underbrace{\langle o(\|h\|), x \rangle}_{= o(\|h\|)}$

$|\langle \mathcal{D}f_x(h), h \rangle| \leq \underbrace{|\mathcal{D}f_x(h)|}_{o(1)} |h| = o(\|h\|)$

$A_x(h) = \langle \mathcal{D}f_x(h), x \rangle + \langle f(x), h \rangle$

est linéaire en h car $\mathcal{D}f_x(h)$ linéaire par hyp.

et $\langle f(x), h\rangle$ linéaire en h car $\langle , \rangle$ bilinéaire.

Donc $F(x+h) = F(x) + A_x(h) + o(\|h\|)$

$A_x(h)$ est continue en x car $Df_x(h)$ continue est

$\langle , \rangle$ continue et $f(x)$ continue.

Donc F est de classe $\mathcal{C}^1$

Exercice 2.1

1. $\frac{\partial F_1}{\partial t}(x,t) = \frac{d}{\partial t} f(x-ct)$

$$= -c \frac{df}{dy}(x-ct)$$

$$\frac{\partial^2 F_1}{\partial t^2}(x,t) = c^2 \frac{d^2 f}{dy^2}(x-ct)$$

$$\frac{\partial F_1}{\partial x}(x,t) = \frac{d}{dy} f(x-ct)$$

$$\frac{\partial^2 F_1}{\partial x^2}(x,t) = \frac{d^2}{dy^2} f(x-ct)$$

d'où $\frac{\partial^2 F_1}{\partial t^2}(x,t) = c^2 \frac{\partial^2 F_1}{\partial t^2}$

idem pour F_2

2. $\tilde{F} = F(\phi(X,Y)) = F\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)$

$$\frac{\partial \tilde{F}}{\partial x}(x,y) = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{1}{2}\frac{\partial F}{\partial u}(x,t) - \frac{1}{2c}\frac{\partial F}{\partial v}(x,y)$$

$$\frac{\partial^2 \tilde{F}}{\partial x \partial y}(x,y) = 0 \cdot \cancel{\frac{\partial F}{\partial u}(u,v)} + \left(\frac{\partial^2 F}{\partial u^2}(u,v)\frac{\partial u}{\partial y}(x,y) + \frac{\partial^2 F}{\partial u \partial v}(u,v)\frac{\partial v}{\partial y}(x,y)\right)$$

$$-\cancel{0\cdot\frac{\partial F}{\partial v}(u,v)} - \frac{\partial^2 F}{\partial v\partial u}(u,v)\frac{\partial u}{\partial y}(x,y) - \frac{\partial^2 F}{\partial v^2}(u,v)\frac{\partial v}{\partial y}(x,y)$$

$$\frac{\partial^2 F}{\partial u^2}(u,v)\frac{1}{2} + \frac{\partial^2 F}{\partial u\partial v}\frac{1}{2c}$$

$$-\frac{\partial^2 F}{\partial v\partial u}\frac{1}{2} - \frac{\partial^2 F}{\partial v^2}\frac{1}{2c}$$

$$\tilde{F}(X,Y) = F\left(\frac{X+Y}{2}, \frac{Y-X}{2}\right)$$

$$\frac{\partial \tilde{F}}{\partial X}(X,Y) = \frac{\partial F}{\partial X}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)$$

$$= \frac{1}{2}\frac{\partial F}{\partial u}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right) - \frac{1}{2c}\frac{\partial F}{\partial v}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)$$

$$\frac{\partial^2 \tilde{F}}{\partial X\partial Y}(X,Y) = \frac{\partial^2 F}{\partial X\partial Y}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)$$

$$= \cancel{\frac{1}{4}\frac{\partial^2 F}{\partial u^2}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)} + \cancel{\frac{1}{4c}\frac{\partial^2 F}{\partial u\partial v}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)}$$

$$\cancel{-\frac{1}{4c}\frac{\partial^2 F}{\partial v\partial u}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)} - \cancel{\frac{1}{4c^2}\frac{\partial^2 F}{\partial v^2}\left(\frac{X+Y}{2}, \frac{Y-X}{2c}\right)}$$

$$=0$$

car F C^2

$$\frac{\partial^2 F}{\partial v^2} = \frac{1}{4}\frac{\partial^2 F}{\partial u^2}$$

b)

2.

Exercice 2.3

$M \longmapsto \det(M)$

1. polynomiale par la formule de déterminant

2. $\dfrac{\partial}{\partial E_{i,j}}(Id) = \dfrac{\det(Id + tE_{i,j}) - \det(Id)}{t} = \begin{cases} \frac{1+t-1}{t} = 1 & \text{si } i=j \\ 0 & \text{si } i \neq j \end{cases}$

Donc $J_{Id} = Id$

D'où $D_{Id}(H) = Tr(H)$

3. Soit $A \in GL_n(\mathbb{R})$

$$\begin{aligned} \det(A+H) &= \det(A(Id + A^{-1}H)) \\ &= \det(A)\det(Id + A^{-1}H) \\ &= \det(A)(\det(Id) + Tr(A^{-1}H) + o(\|A^{-1}H\|)) \\ &= \det(A)(\det(Id) + Tr(\tfrac{{}^t Com(A)}{\det(A)}H) + o(\|A^{-1}H\|)) \\ &= \det(A)(\det(Id) + \tfrac{1}{\det(A)} Tr({}^t Com(A)H)) + o(\|A^{-1}H\|) \\ &= \det(A) + Tr({}^t Com(A)H) + o(\|A^{-1}H\|) \end{aligned}$$

$H \mapsto Tr({}^t Com(A) \cdot H)$ est linéaire donc $Df_A(H) = Tr({}^t Com(A)H)$

4) $GL_n(\mathbb{R}) = f^{-1}(\mathbb{R}^*)$ l'image réciproque d'un ouvert par une fonction continue.

Donc $GL_n(\mathbb{R})$ est ouvert. Montrons que $GL_n(\mathbb{R})$ est dense dans $M_n(\mathbb{R})$. Soit $M_0 \in M_n(\mathbb{R})$ de $\det(M_0)=0$

$g(z) = \det(M_0 + z Id) = z^n(\frac{1}{z}M_0 + Id) \underset{z \to +\infty}{\sim} z^n \cdot 1 = z^n$

donc $P_{M_0}(z)$ admet au plus n racines

donc si on prends $0 < z_0 < \min_z \{z : g(z) = 0\}$

On a $\forall\ \varepsilon \in [0, \varepsilon_0]$ $\quad \det(M_0 + \varepsilon I_d) \neq 0$

d'où $GL_n(\mathbb{R})$ est dense dans $\mathbb{R}$.

On prends $(A_n)_{n\in\mathbb{N}}$ d'éléments de $GL_n(\mathbb{R})$

tq $A_n \xrightarrow[n\to+\infty]{} M_0$

Alors $Df A_n \longrightarrow Df M_0$

d'où $Df A_n(H) = Tr({}^t\mathrm{com} A \cdot H) \longrightarrow Tr({}^t\mathrm{Com}\, M_0 \cdot H)$

5.] $f : M \longmapsto M^{-1}$ Or $M \in GL_n(\mathbb{R})$ donc

$\det(M) \neq 0$ de plus $\det(M)$ polynomiale donc C^∞
d'où $\frac{1}{\det M}$ l'est aussi.

$M^{-1} = \frac{1}{\det M}\, {}^t\mathrm{Com}\, M$

Posons $S = \{(M,N) \in GL_n(\mathbb{R})^2 \mid g(M,N) = MN - I = 0\}$
de classe C^∞

Soit $M_0 \in GL_n(\mathbb{R})$ et $N_0 = M^{-1}$

donc $g(M_0, N_0) = 0$

$D_N F(M_0, N_0) = M_0$ inversible car $M_0 \in GL_n(\mathbb{R})$

$D_N F(M_0, N_0)(H) = M_0 H$ est la différentielle de $g_{\bar{x},\cdot}$

D'où $\exists\ \varphi : GL_n(\mathbb{R}) \longrightarrow M_n(\mathbb{R})$

tq $g(M, \varphi(M)) = M\varphi(M) - I = 0$

d'où $\varphi(M) = M^{-1}$. Le théorème des fonction implicites donne que φ est de classe C^∞

Exercice 2.4 soit $k \in \mathbb{N}^*$

$$f : \mathcal{M}_n(\mathbb{R}) \longrightarrow \mathcal{M}_n(\mathbb{R})$$
$$M \longmapsto M^k$$

$$f(M)_{j,k} = \begin{cases} \sum_{i=0}^{n} M^{\frac{k}{2}}_{i,j} \, M^{\frac{k}{2}}_{k,i} & \text{si } k \text{ pair} \\ \sum_{i=0}^{n} M^{k-1}_{i,j} \, M_{k,i} & \text{si } k \text{ impair} \end{cases}$$

par récurrence on obtient que pour chaque élément d'entrée de f, M^k est polynomiale donc de classe $\mathcal{C}^\infty$

$$f(A+H) - f(A) = (A+H)^k - A^k$$

$$d(MMM) = (dM)MM + M\overbrace{d(MM)}^{d(M^2)}$$

$$d(M^3) = (dM)M^2 + M\,d(M^2)$$
$$= (dM)M^2 + M(dM)M + M^2 dM$$

$$d(M^4) = (dM)M^3 + M(dM^3)$$
$$= (dM)M^3 + M(dM)M^2 + M^2(dM)M + M^3(dM)$$

AHA

$$d(M^k) = \sum_{i=0}^{k-1} M^i (dM) M^{k-i-1}$$

$$f(A+H) - f(A) = (A+H)^k - A^k$$
$$= A^k + \sum_{i=1}^{k} A^{k-i-1} H A^i - A^k \quad + o(\|H\|)$$
$$= \sum_{i=1}^{k} \underbrace{A^{k-i} \, H A^{i-1}}$$

chaque terme linéaire en H

$$\text{donc} \quad d_A f(H) = \sum_{i=1}^{k} A^{k-i} \, H A^{i-1}$$

3) Soit $k = 2$

$$f(A+H) - f(A) - d_A f(H) = (A+H)^2 - A^2 - AH - HA$$

$$\frac{1}{2} d_A^2 f(H,H) = H^2$$

Donc $d_A^2 f(H,H) = 2H^2$

$$d_A^3 f(H,H,H) = f(A+H) - f(A) - d_A f(H) - \frac{1}{2} d_A^2 f(H,H) = 0$$

Exercice 2.5

$\alpha \in]0,1[\quad f(K,L) = \lambda K^\alpha L^{1-\alpha}$

1. Pour $(K,L) \in \mathbb{R}_+ \times \mathbb{R}_+$

$$\frac{\partial f}{\partial K}(K,L) = \alpha \lambda L^{1-\alpha} K^{\alpha-1}$$

$$\frac{\partial f}{\partial L}(K,L) = (1-\alpha)\lambda K^\alpha L^{-\alpha}$$

$$\frac{\partial^2 f}{\partial L \partial K}(K,L) = (1-\alpha)\alpha\lambda L^{-\alpha} K^{\alpha-1}$$

$$\frac{\partial^2}{\partial K \partial L}(K,L) = \alpha(1-\alpha)\lambda K^{\alpha-1} L^{-\alpha}$$

D'où $\frac{\partial f}{\partial L \partial K}(K,L) = \frac{\partial^2}{\partial K \partial L}(K,L)$ et donc le théorème de Schwartz est vérifié.

2)

Exercice 2-6

1. $$g(x) = \left(\frac{\partial f}{\partial x_1} \middle| \frac{\partial f}{\partial x_2} \middle| \quad \cdots \quad \middle| \frac{\partial f}{\partial x_n} \right)(x) \cdot h$$

$$g(x) = \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}$$

pour $j \in [\![1, n]\!]$

$$\frac{\partial g}{\partial x_j}(x) = \sum_{i=1}^{n} h_i \frac{\partial^2 f}{\partial x_j \partial x_i}$$

2) $$D_x g(h') = \sum_{j=1}^{n} h'_j \sum_{i=1}^{n} h_i \frac{\partial^2 f}{\partial x_j x_i} = D^2_x f(h, h')$$

$$D_x g(h') = (D_x^2 f(h))(h')$$

3)

$$D_x F(k) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$D_x(D_x f(h))(k) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$D_x^2 f(k, h) = (D_x F(k))(h)$$

$$D_x F : \mathbb{R}^n \longrightarrow \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$$

$$D_x F \in \mathcal{L}(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m))$$

Exercice 2.7

$$D_x(g \circ f)\, h = (D_x f \cdot D_{f(x)} g)\, h$$

$$D^2_x(g \circ f)(h, h') = (D^2_x f\, D_{f(x)} g + D_x f\, D^2_{f(x)} g)(h, h')$$

$$\frac{\partial z}{\partial x_i}(x) = \sum_{k=1} \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_k}(f(x))$$

$$\frac{\partial^2 z}{\partial x_j \partial x_i}(x) = \sum_{k=1} \frac{\partial}{\partial x_j}\left(\frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_k}(f(x)) \right)$$

$$= \sum_{k=1}^{m} \left(\frac{\partial^2 f}{\partial x_j x_i}(x) \frac{\partial g}{\partial x_k}(f(x)) + \frac{\partial f}{\partial x_i}(x) \sum_{k'=1}^{m} \frac{\partial f}{\partial x_j}(x) \frac{\partial^2 g}{\partial x_{k'} \partial x_k}(f(x)) \right)$$

$$\underbrace{D_x(g \circ f)}_{= 3} = \sum_{q=1}^{m} h_q \frac{\partial 3}{\partial x_q}(x) = \sum_{q=1}^{m} h_q \sum_{k=1}^{m} \frac{\partial f}{\partial x_i}(x) \frac{\partial g}{\partial x_k}(f(x))$$

$$D_x^2(g \circ f) = \sum_{i=1}^{m} \sum_{j=1}^{m} h_i h_j \left(\sum_{k=1}^{m} \left(\frac{\partial^2 f}{\partial x_j x_i}(x) \frac{\partial g}{\partial x_k}(f(x)) + \frac{\partial f}{\partial x_i}(x) \sum_{k'=1}^{m} \frac{\partial f}{\partial x_j}(x) \frac{\partial^2 g}{\partial x_{k'} \partial x_k}(f(x)) \right) \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{m} h_i h'_j \left(\sum_{k=1}^{m} \frac{\partial^2 f}{\partial x_j x_i}(x) \frac{\partial g}{\partial x_k}(f(x)) + \frac{\partial f}{\partial x_i}(x) \sum_{k'=1}^{m} \frac{\partial f}{\partial x_j}(x) \frac{\partial^2 g}{\partial x_{k'} \partial x_k}(f(x)) \right)$$

$$= h^T \left(D_x^2 f \left(D_{f(x)} g + D_x f\, D^2_{f(x)} g\, D_x f \right) \right) h$$

$$\frac{\partial (g_a \circ f)}{\partial x_i}(x) = \sum_{k=1}^{m} \frac{\partial f_k}{\partial x_i}(x) \frac{\partial g_a}{\partial x_k}(f(x))$$

$$\frac{\partial^2 (g_a \circ f)}{\partial x_j \partial x_i}(x) = \sum_{k=1}^{m} \left(\frac{\partial f_k}{\partial x_j \partial x_i}(x) \frac{\partial g_a}{\partial x_k}(f(x)) + \frac{\partial f_k}{\partial x_i}(x) \left(\sum_{k'=1}^{m} \frac{\partial f_{k'}}{\partial x_j}(x) \frac{\partial^2 g_a}{\partial x_{k'} \partial x_k} \right.\right.$$

$$= \sum_{k=1}^{m} \frac{\partial^2 f_k}{\partial x_j x_i}(x) \frac{\partial g_a}{\partial x_k}(f(x)) + \sum_{k,z=1}^{m} \frac{\partial f_k}{\partial x_i}(x) \frac{\partial^2 g_z}{\partial x_z \partial x_k}(f(x)) \frac{\partial f_z}{\partial x_j}$$

$$D_x^2\, g_a \circ f(h, h') = \sum_{i,j=1}^{m} h_i h_j \frac{\partial^2 (g_a \circ f)}{\partial x_j x_i}$$

$$= \sum_{i,j=1}^{m} h_i h_j \sum_{k=1}^{m} \frac{\partial^2 f_k}{\partial x_j x_i}(x) \frac{\partial g_a}{\partial x_k}(f(x))$$

$$+ \sum_{i,j=1}^{m} h_i h_j \sum_{k,z=1}^{m} \frac{\partial f_k}{\partial x_i}(x) \frac{\partial^2 g_z}{\partial x_z \partial x_k}(f(x)) \frac{\partial f_z}{\partial x_j}$$

$$= \sum_{k=1}^{m} \sum_{i,j=1}^{m} \frac{\partial g_a}{\partial x_k}(f(x)) \frac{\partial^2 f_k}{\partial x_j x_i}(x) h_i h'_j + \sum_{k,z=1}^{m} \sum_{i,j=1}^{m} \frac{\partial^2 g_z}{\partial x_z \partial x_k}(f(x)) \frac{\partial f_k}{\partial x_i}(x) \frac{\partial f_z}{\partial x_j}(x) h_i h_j$$

$$= \sum_{k=1}^{m} \frac{\partial g_n}{\partial x_k}(f(x))\, D_x^2 f_k(h,h')$$
$$= D_{f(x)} g_n \left(D_x^2 f\,(h,h') \right) + D^2_{f(x)} g_n \left(D_x f(h),\, D_x f(h') \right)$$

Exercice 1.5

$f : \mathbb{R}^n \to \mathbb{R}$ $\qquad f(tx) = t^2 f(x)$

homogène de degré 2

1. Soit $t>0$. Supposons que

$f(tx) = t^2 f(x)$ et admet des dérivées partielles

$$\frac{\partial f}{\partial x_i}(x) = \lim_{t \to 0} \frac{f(x+te_i) - f(x)}{t} = \frac{f(x+(tx^{(i)})) - f(x)}{t}$$

$\frac{1}{t}(f(x))$

$t^{-1} f(x)$

$= f(t^{-\frac{1}{2}} x)$

$$f(x) = \frac{f(tx)}{t^2}$$

$$f(x+te_i) = f(t(\tfrac{1}{t}x + e_i)) = t^2 f(\tfrac{1}{t}x + e_i)$$

$$\lim_{t \to 0} t^{2-1} f(\tfrac{1}{t}x + e_i) - f(t^{-\frac{1}{2}} x)$$

$$\nabla f(x) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(x) \\ \vdots \\ \frac{\partial f}{\partial x_n}(x) \end{pmatrix} = \Big($$

$$\langle \nabla f(x), x \rangle = x^{(1)} \frac{\partial f}{\partial x_1}(x) + \dots + x^{(n)} \frac{\partial f}{\partial x_n}(x)$$

$$g(t) = \frac{f(x+te_i) - f(x)}{t}$$

$$g(t) = f(tx) \qquad \frac{dg}{dt}(t) = \frac{d}{dt} f(tx) = \sum \underbrace{\frac{d\,tx}{dt}(t)}_{x} \frac{\partial f}{\partial y_k}(tx) = \langle \nabla f(tx), x \rangle$$
$$= t^2 f(x)$$

$$\frac{d}{dt} t^2 f(x) = 2 t^{2-1} f(x)$$

en $t=1$ on a $\quad g'(1) = \langle \nabla f(x), x \rangle = 2 f(x)$

2) Supposons que f admet des dérivées partielles et

$\langle \nabla f(x), x \rangle = 2 f(x)$

a) Soit $x \in \mathbb{R}^n$

$F(t) = f(tx) - t^2 f(x)$

$$\frac{df}{dt}(tx) = \sum_{k=1}^{n} \frac{\partial f}{\partial k}(tx) x^{(k)} = \langle \nabla f(tx), x \rangle = \frac{1}{t} \langle \nabla f(tx), tx \rangle = \frac{1}{t} 2 f(tx)$$

$$\frac{d}{dt} t^2 f(x) = 2 f(x) t^{2-1}$$

$$F'(t) = \frac{1}{t} 2 f(tx) - 2 f(x) t^{2-1} = \frac{2}{t} \big(\underbrace{f(tx) - t^2 f(x)}_{= F(t)} \big) = \frac{2}{t} F(t)$$

b) $F(1) = f(x) - f(x) = 0$

c) On a $\forall i \in \mathbb{N}^* \quad F^{(i)}(1) = 0$

d'où $\forall t$ F est constante, d'après b)
$\forall t \quad F(t) = 0 \quad$ i.e

$\forall t \quad f(tx) = t^2 f(x) \quad$ d'où l'homogénité de degré 2.

3) Par l'homogénité on obtient l'identité d'euler,
par l'identité d'euler on a

$F(t) = f(tx) - t^2 f(x)$ avec $F'(t) = \frac{2}{t} F(t)$

$$\frac{\partial f}{\partial x_i}(tx) = t^{\cancel{2-1}} \cancel{\frac{\partial f}{\partial x_i}}(x)$$

$$= \frac{f(tx + h e_i) - f(tx)}{h} = \frac{f(t(x + \frac{h}{t} e_i)) - f(tx)}{h}$$

$$= \frac{t^2 (f(x + \frac{h}{t} e_i) - f(x)}{h}$$

$$= t^2 \frac{\frac{\partial f}{\partial x_i}(x)}{t}$$

$$= t^{2-1} \frac{\partial f}{\partial x_i}(x)$$