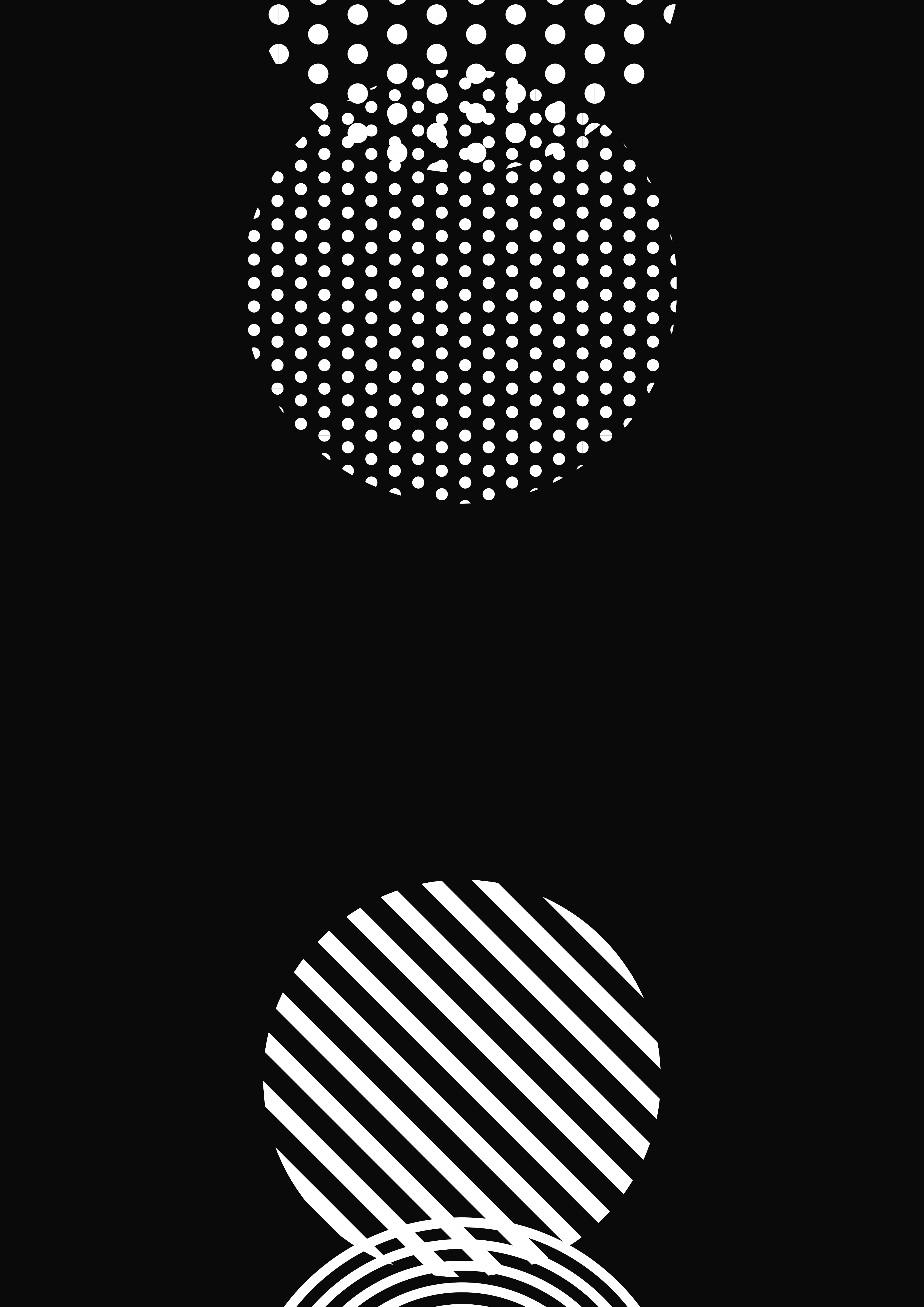

$$p(\lambda) = \int_0^1 |f-\lambda g|^2 dx$$

$$p(\lambda) = 0 \iff \int_0^1 |f-\lambda g|^2 dx$$

$$\iff \forall x \in [0,1] \quad |f(x) - \lambda g(x)|^2 = 0 \quad \text{car} \quad \text{positive}$$

$$\iff \forall x \in [0,1] \quad f(x) - \lambda g(x) = 0$$

$$\iff \forall x \in [0,1] \quad f(x) = \lambda g(x)$$

$$f^2 = \lambda fg$$

$$g^2 =$$

$$\int_0^1 f^2 dx - 2\lambda \int_0^1 fg\,dx \quad + \lambda^2 \int_0^1 g^2 dx = 0$$

$$\mathcal{D} = 4\left(\int_0^1 fg\right)^2 - 4\int_0^1 f^2 dx \int_0^1 g^2 dx \leq 0$$

$$\Rightarrow \left(\int_0^1 fg\right)^2 \leq \int_0^1 f^2 dx \int_0^1 g^2 dx$$

$$\Rightarrow$$

10) $$|f_n - f|^2 = |(f_n - f) - (1-1)|$$

$\boxed{1.4}$

$$\Rightarrow \int_0^\infty |\phi \sin(x)|\, dx \quad cv$$

i.e $\exists l \in \mathbb{R}$ $\int_0^\infty |\phi(x)\sin(x)| dx = l$

$$= \int_0^\infty |\phi(x)||\sin(x)| dx \geq \left|\int_0^\infty \phi(x)\sin(x) dx\right|$$

donc $\int_0^\infty \phi(x) \sin(x)\, dx$ cv

$$\leq \int_0^\infty \phi(x)|\sin(x)|\, dx$$

Notons $u_n = \int_{n\pi}^{(n+1)\pi} \phi(x) \sin(x) dx$ alors

$\forall n \in \mathbb{N}$ $\quad u_{2n} \geq 0$ et $u_{2n+1} \leq 0$

$$|u_n| = \int_{n\pi}^{(n+1)\pi} \phi(x)|\sin(x)| dx \geq |u_{n+1}| = \int_{(n+1)\pi}^{(n+2)\pi} \phi(x)|\sin(x)|\, dx$$

car ϕ décroissante de plus $\phi(x) \longrightarrow 0$

donc soit $\varepsilon > 0$ $\quad \exists x' \in \mathbb{R}$ tq $\quad \forall x \geq x'$ $\phi(x) \leq \varepsilon$

donc $\exists N' \in \mathbb{N}$ tq $\forall n \geq N'$, $n\pi \geq x'$

et donc $\int_{n\pi}^{(n+1)\pi} \phi(x)|\sin(x)|dx \leq \varepsilon \int_{n\pi}^{(n+1)\pi} |\sin(x)|dx = \begin{cases} -\cos(x) + \cos(x) = 2 & \text{si } n \text{ pair} \\ \cos(x) - \cos(x) = 2 \end{cases}$

$= 2\varepsilon$

Donc $\int_{n\pi}^{(n+1)\pi} \phi(x)|\sin(x)|dx \leq 2\varepsilon$ donc cv vers 0

Alors le critère les séries alternées s'applique d'où $\sum u_n$ cv.

$\forall n \in \mathbb{N}$ $\quad \forall x \in [n\pi, (n+1)\pi]$ $\quad \phi((n+1)\pi) \leq \phi(x) \leq \phi(n\pi)$

D'où $\int_{n\pi}^{(n+1)\pi} \phi(x)\sin(x)dx \geq \phi((n+1)\pi) \int_{n\pi}^{(n+1)\pi} |\sin(x)|dx$

$= 2\phi((n+1)\pi)$

D'où $\int_0^{\infty} \phi(x)\sin(x)dx \geq 2\sum_{n=1}^{+\infty} \phi((n+1)\pi)$

$\parallel$

$\sum_{n=1}^{+\infty} \int_{n\pi}^{(n+1)\pi} \phi(x)\sin(x)dx$ $\quad$ donc $\sum \phi((n+1)\pi)$ cv

$\int_n^{(n+1)\pi} \phi(x)dx \leq \phi(n\pi)\pi$

Donc $\sum_{n=1}^{+\infty} \int_n^{(n+1)\pi} \phi(x)dx \leq \pi \sum_{n=1}^{+\infty} \phi(n\pi)$ cv

donc $\int_0^{+\infty} \phi(x)dx$ cv.

$\boxed{\Leftarrow}$ Supposons que $\int_0^{+\infty} \phi(x)dx$ converge

On a $\int_0^{+\infty} |\phi(x)\sin(x)|dx = \int_0^{+\infty} \phi(x)|\sin(x)|dx$

$\leq \int_0^{+\infty} \phi(x)dx$ car $|\sin(x)| \leq 1$ $\forall x \in \mathbb{R}$

D'où $\int_0^{+\infty} \phi(x)\sin(x)dx$ cv. absolument.