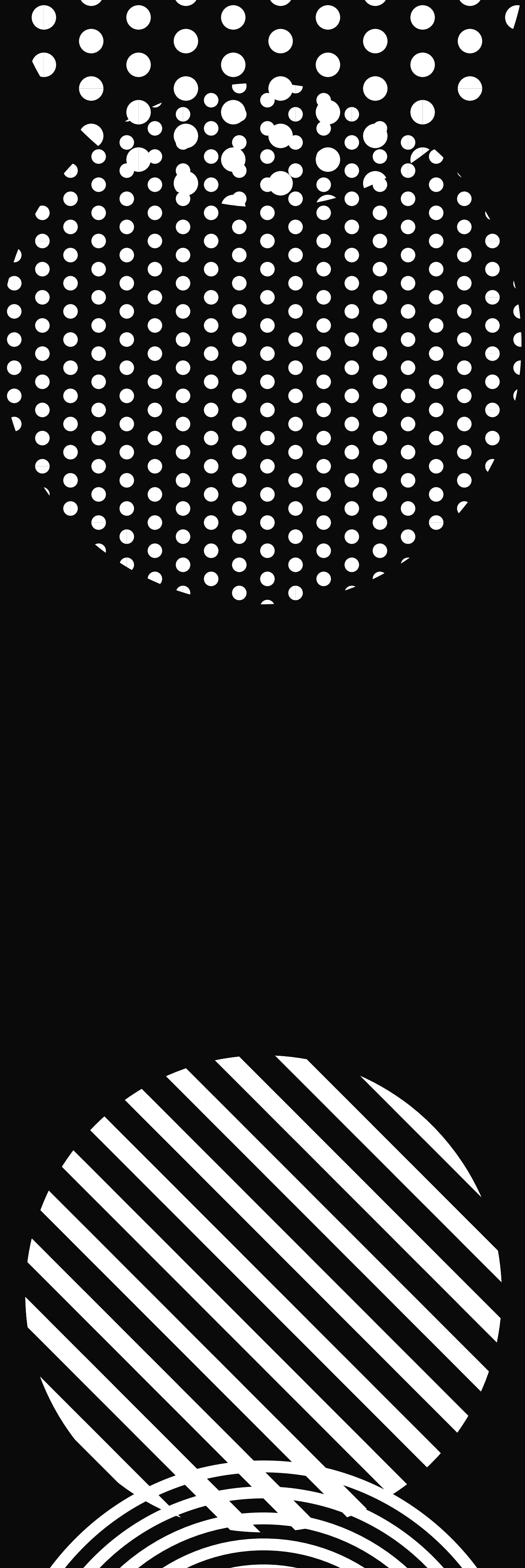

Exercice 9.1

(a) $$\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\frac{1}{(x^2+1)(y^2+1)}\,dx\right)dy$$
$$= \int_{\mathbb{R}} \frac{1}{y^2+1}\,[\arctan(x)]_{\mathbb{R}}\,dy$$
$$= \int_{\mathbb{R}} \pi \,\frac{1}{y^2+1}\,dy = \pi^2$$

(b) $$\iint_{\mathbb{R}\,\mathbb{R}} \frac{1+x^2}{1+y^2}\,dy\,dx = \int_{\mathbb{R}} \pi\,(1+x^2)\,dx = +\infty$$

donc n'est pas intégrable.

(c) $$\frac{|\cos(xy)|}{(x^2+1)(y^2+1)} \leq \frac{1}{(x^2+1)(y^2+1)}$$ qui est intégrable d'après (a)

(d) $$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{x-1\leq y\leq x+1\}}\,|x,y|\,dx\,dy$$
$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \mathbb{1}_{\{y\in[x-1,x+1]\}}\,dy\,dx$$
$$= \int_{\mathbb{R}}\int_{[x-1,x+1]} 1\,dy\,dx$$
$$= \int_{\mathbb{R}} [t]_{x-1}^{x+1}\,dx$$
$$= \int_{\mathbb{R}} x+1-x+1\,dx = \int_{\mathbb{R}} 2\,dx = +\infty$$

donc n'est pas intégrable.

1. (a) <u>proprement</u> !

$$\int_{\mathbb{R}^2} \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy \overset{\text{Fubini Tonelli}}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{1}{(x^2+1)(y^2+1)}\,dx\,dy$$
$$= \int_{\mathbb{R}} \frac{1}{y^2+1}\left(\int_{\mathbb{R}} \frac{1}{x^2+1}\,dx\right)dy$$
$$= \left(\int_{\mathbb{R}} \frac{1}{(x^2+1)}\,dx\right)\left(\int_{\mathbb{R}} \frac{1}{y^2+1}\,dy\right) < +\infty$$

$$\int_{\mathbb{R}} \frac{1}{y^2+1}\,dz \overset{\text{TCM}}{=} \lim_{a\to+\infty} \int_{[-a,a]} \frac{1}{z^2+1}\,dz$$
$$\underset{\text{comparaison Riemann}}{=} \lim_{a\to+\infty} \int_{-a}^{a} \frac{1}{z^2+1}\,dz = \pi$$

2. Contre-exemple : $\mathbb{1}$ (d)

Exercice 9.2

1. $\int_{[0,\frac{\pi}{2}]} \mathbb{1} \, d\lambda_2(x,y) = \lambda_2([0,\frac{\pi}{2}]^2) = \frac{\pi}{2} \cdot \frac{\pi}{2} = \frac{\pi^2}{4}$

2. $f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x,y) = \cos(x+y) \, \mathbb{1}_{[0,\frac{\pi}{2}]^2}(x,y)$

intégrable car $\cos(x+y)$ continue sur un compact $[0,\frac{\pi}{2}]^2$

3. $\int_{\mathbb{R}^2} \cos(x+y) \, dx\,dy$

$\mathbb{R}^2 \to \mathbb{R}^2$

$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} x+y \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$$= \int_{[0,\frac{\pi}{2}]} \int_{[v, v+\frac{\pi}{2}]} \cos(u) \, du \, dv$$

$$= \int_{[0,\frac{\pi}{2}]} \sin(v+\tfrac{\pi}{2}) - \sin(v) \, dv$$

$$= \sin(\pi) - \sin(\tfrac{\pi}{2}) - \sin(\tfrac{\pi}{2}) + \sin(0) = -2\sin(\tfrac{\pi}{2}) = -2$$