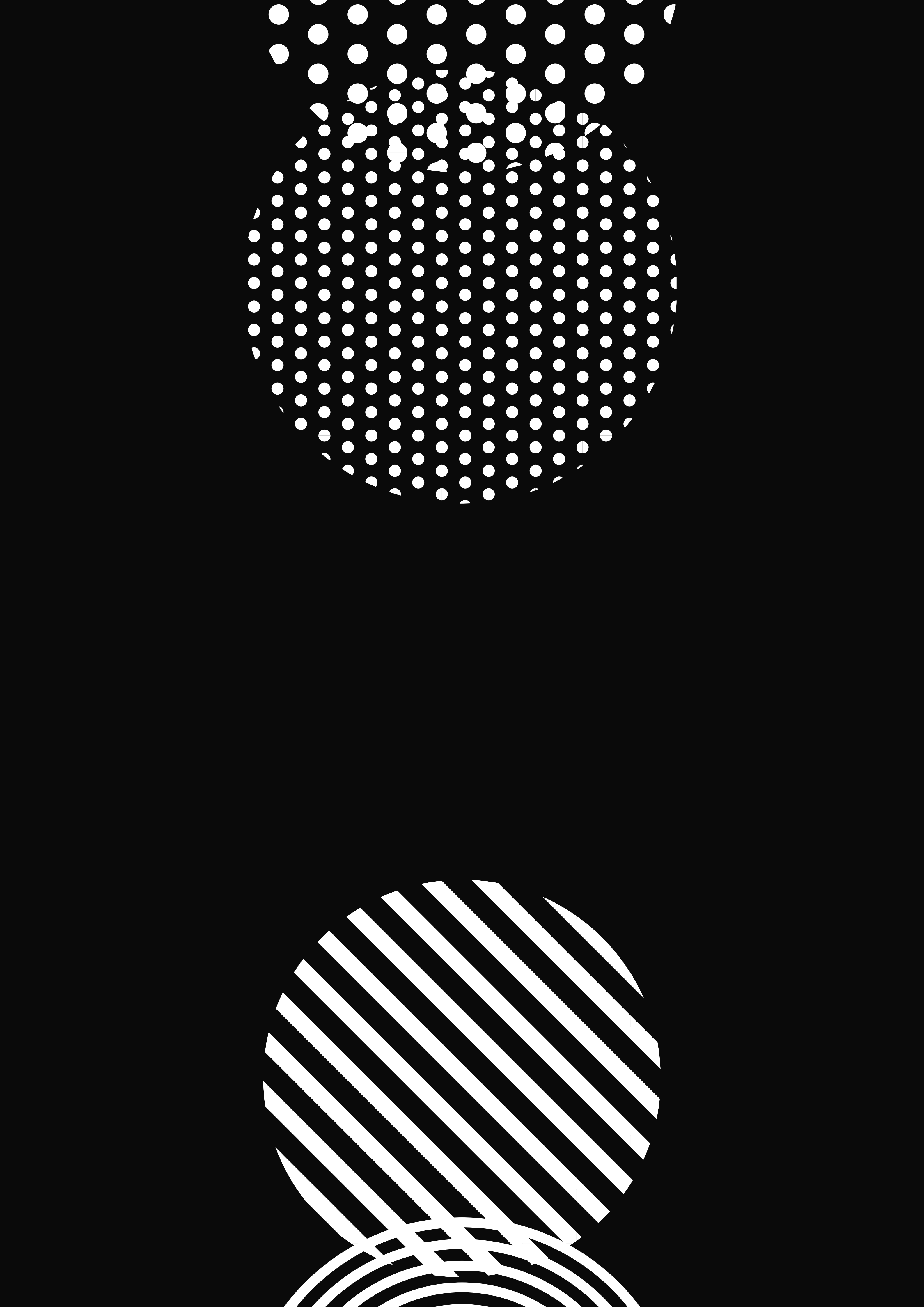

Exercise 2.1.15

$$\bigcap E = \left(\bigcup E^c\right)^c$$

a) $\limsup E_k = \bigcap \bigcup E_k = \left(\bigcup\left(\bigcup E_k\right)^c\right)^c$

Let $x \in \limsup\limits_{k\to+\infty} E_k$

i.e $x \in \bigcap\limits_{j=1}^{\infty} \bigcup\limits_{k=j}^{\infty} E_k$ thus

$x \in \bigcup\limits_{k=j}^{\infty} E_k \quad \forall j \in \mathbb{N}$

Let $j \in \mathbb{N}$. We suppose that $x \in \bigcup\limits_{k=j}^{\infty} E_k$,

hence there exists k_1 such that $x \in E_{k_1}$.

Let now $j = k_1+1$. As $x \in \limsup E_k$ thus $x \in \bigcup\limits_{k=k_1+1} E_k$ there exist k_2 such that

$x \in E_{k_2}$. We can repeat the process infinitely many times and obtain the infinite set of $K = \{k_1, k_2, \dots\}$ such that $\forall k \in K$, $x \in E_k$.

Consequently, if $x \in \limsup E_k$ then x belong to infinitely many E_k.

b) $\liminf E_k = \bigcup\limits_{j\in\mathbb{N}} \bigcap\limits_{k\geq j} E_k$

Let $x \in \bigcup\limits_{j=1}^{+\infty} \bigcap\limits_{k=j}^{+\infty} E_k$

Let's denote $U_j = \bigcap\limits_{k=j}^{+\infty} E_k$

then $x \in \bigcup\limits_{j=1}^{+\infty} U_j$ that means that there exists

such j_0 such that $x \in U_{j_0}$

where $U_{j_0} = \bigcap\limits_{k=j_0}^{+\infty} E_k$ i.e $x \in E_k \quad \forall k \geq j_0$

Excercice 2.1.16

Proof for lim inf E_k

Let's start with the proof of the fact that

$$\lambda^*\left(\bigcap_{k=j}^{+\infty} E_k\right) \leq \lambda^*(E_q) \quad \forall\, q \geq k$$

Let $k \in \mathbb{N}$ and $\{Q_n\}_{n\in\mathbb{N}}$ a cover of E_k

The hypothesis $\sum |E_k| < +\infty$ gives us the fact that

$|E_k| < +\infty$ is bounded

Also, this hypothesis gives us that $\lim_{k\to+\infty} |E_k| = 0$ due to the convergence of the $\sum |E_k|$

Thus $\lambda^*\left(\bigcap_{k=j}^{+\infty} E_k\right) \leq 0$ as the outer measure is positive,

then $\lambda^*\left(\bigcap_{k=j}^{+\infty} E_k\right) = 0$

$$\lambda^*\left(\bigcup_{k\geq 1}\bigcap_{k=j}^{+\infty} E_k\right) \leq \sum_{j=1}^{+\infty} \lambda^*\left(\bigcap_{k=j}^{+\infty} E_k\right) = \sum_{j=1}^{+\infty} 0 = 0$$

Proof for lim sup E_k

easy by sum convergence after some N

$\sum_{n=N}^{+\infty} |E_k| < \varepsilon$ then cv. to 0

Excercise 2.2.30

U open and f cont. on U

then for all $x \in U$ for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $y \in B(x,\delta) \Rightarrow |f(x) - f(y)| < \varepsilon$

Let $m = \operatorname{ess\,sup} f$ and $M = \sup f$

Let's suppose that $m \neq M$ i.e. $m < M$ that is to say there exists a set Z of measure 0 such that for all $x \in Z$ $f(x) > m$

Let $\varepsilon > 0$

We construct a sequence $(x_n)_{n\in\mathbb{N}}$ such that

$f(x_n) \xrightarrow[n\to+\infty]{} u$ by continuity of f

$\forall n \in \mathbb{N} \quad \exists r>0 \quad \forall y \in B(x_n, r) \quad |f(x) - f(y)| < \varepsilon$

By taking ε arbitrary small we obtain an open ball $U_B \subset Z$ but $|U_B|_e > 0$

and $|U_B| < |Z| = 0$

Contradiction, then $m = u$

Exercise 2.3.6

a) Let $Q \subset \mathbb{R}^m$ be a box i.e $Q = \bigtimes_{i=1}^{m} [a_i, b_i]$
and $R \subset \mathbb{R}^n$ be a box i.e $R = \bigtimes_{i=1}^{n} [a_i', b_i']$

then $Q \times R = \bigtimes_{i=1}^{n+m} [a_i, b_i)$ that is a box in $\mathbb{R}^{n+m}$

then $|Q \times R| = \prod_{i=1}^{m} [a_i, b_i] \cdot \prod_{i=1}^{n} [a_i', b_i'] = |Q| \cdot |R|$

b) $U \subset \mathbb{R}^m$ open and $V \subset \mathbb{R}^n$ open
than $\forall u \in U, \exists r''>0 \quad B(u, r'') \subset U$
$\forall v \in V, \exists r'>0 \quad B(v, r') \subset V$ we take $r = \min(r'', r')$

and obtain $B(u, r) \subset U$
$B(v, r) \subset V$

$U \times V = \{(u,v) : u \in U, v \in V\}$

$B((u,v), r) \subset U \times V$

It is evident that $|U \times V| \leq |U||V|$
then we are left to show that $|U \times V| \geq |U||V|$

$|U \times V|_e = \inf\{\prod_{i=1}^{n+m} [a_i, b_i] : \bigtimes_{i=1}^{m+n} [a_i, b_i] \supset U \times V\}$

$= \inf\left\{\prod_{i=1}^{n+m} [a_i, b_i] : \forall (u,v) : u \in U, v \in V \quad (u,v) \in \bigtimes_{i=1}^{m+n} [a_i, b_i]\right\}$

then $|U \times V|_e \geq |U||V|$

Exercise 4.1.10

=> Suppose that $\int_E f = 0 \geq \int_{\{f>0\}} f \geq \int$

Let $\varepsilon > 0$

$\int_E f = 0 \geq \int_{\{f>\varepsilon\}} f \geq \int_{\{f>\varepsilon\}} \varepsilon = \varepsilon |\{f>0\}|$

Thus $\varepsilon |\{f>\varepsilon\}| \leq \int_E f = 0$ and $\varepsilon \neq 0$ then $|\{f>\varepsilon\}| = 0$. As $\varepsilon > 0$ was arbitrary, we make it go to 0 and obtain then $|\{f>0\}| = 0$. Consequently, f is 0 almost everywhere.

[<=] Suppose that $f = 0$ a.e.
It is equivalent to $|\{f>0\}| = 0$ (f positive)

Let $\varepsilon = \frac{1}{n}$

$\frac{1}{\varepsilon} \int_E f \geq \frac{1}{\varepsilon} \int_{\{f>\varepsilon\}} f \geq \frac{1}{\varepsilon} \int_{\{f>\varepsilon\}} \varepsilon \geq |\{f>\varepsilon\}|$

$\Rightarrow n \int_E f \geq |\{f > \frac{1}{n}\}|$

$\int_E f = \int_{\{f>0\}} f = \sup\{ \sum_{i=1}^{N} c_i |E_i| = \int_{\{f>0\}} \varphi : \varphi \leq f$ and φ simple$\}$
$(E_i)_{i=1}^{N}$ is a partition of $\{f>0\}$)

By hypotheses, $|\{f>0\}| = 0$ then $\forall i \in [1,N]$ $|E_i| = 0$, then
$\int_{\{f>0\}} f = 0 = \int_E f$

Exercise 4.5-3

Suppose that $f_n \xrightarrow{a.e} f$ and there exist g such that
$|f_n(x)| \leq g$ a.e $\forall n \in \mathbb{N}$

then let $x \in \mathbb{R}$ $|f(x) - f_n(x)| \leq |f(x)| + |f_n(x)| \leq 2g$ then

$2g - |f - f_n| \geq 0$ a.e

We know that $\liminf_{n \to +\infty} 2g - |f - f_n| = 2g$ as $f_n \to f$ a.e

then $\int 2g = 2\int g = \int \liminf 2g - |f - f_n|$

$\leq \liminf \int 2g - |f - f_n|$

$= \liminf \int 2g - \int |f - f_n| = \int 2g - \limsup \int |f - f_n|$

$\int 2g - \limsup \int |f-f_n| \leq \int 2g \leq \int 2g - \limsup \int |f-f_n| \leq \limsup \int 2g - |f-f_n|$

$= \int 2g - \liminf \int |f-f_n|$

$\leq \int 2g$ car $|f-f_n|$ positive

$0 \leq -\limsup \int |f-f_n| \leq -\int \liminf |f-f_n| \leq 0$

donc $\lim \int |f-f_n| = 0$ i.e $\|f-f_n\|_1 \xrightarrow[n\to+\infty]{} 0$

thus $\int f_n \to \int f$

Exercise 4.5.5