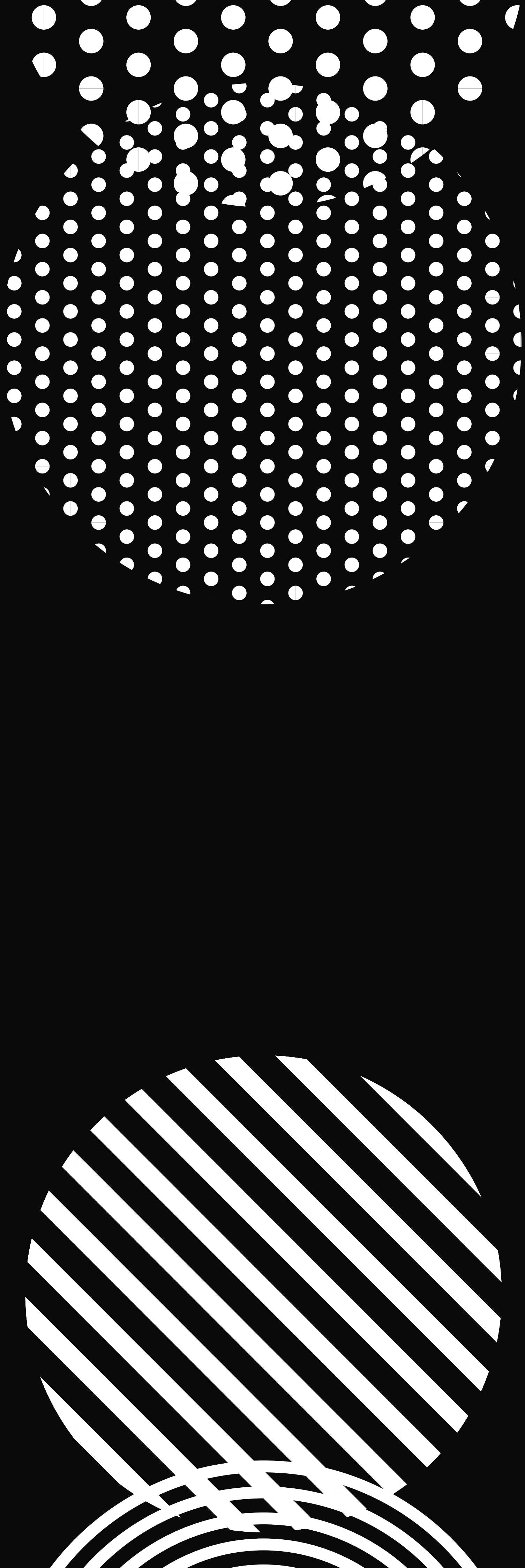

Exercice 2.5

$$\begin{cases} y_1' = y_1 + 8y_2 + e^t \\ y_2' = 2y_1 + y_2 + e^{-3t} \end{cases}$$

1. $Y' = AY + B$

$$= \begin{pmatrix} 1 & 8 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^t \\ e^{-3t} \end{pmatrix}$$

$$\begin{pmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 - 16 \qquad = 0$$

$$(1-\lambda)^2 = 16$$

$$\Leftrightarrow 1-\lambda = 4 \quad \text{ou} \quad 1-\lambda = -4$$

$$\Leftrightarrow \lambda = -3 \qquad \lambda = 5$$

$$\begin{pmatrix} 4 & 8 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{matrix} 4x_1 + 8x_2 \\ 2x_1 + 4x_2 \end{matrix} \qquad \begin{pmatrix} -2 \\ 1 \end{pmatrix} x_1$$

$$x_1 = -2x_2$$

$$\begin{pmatrix} -4 & 8 \\ 2 & -4 \end{pmatrix} \quad \begin{matrix} -4x_1 + 8x_2 \\ 2x_1 - 4x_2 \end{matrix}$$

$$x_1 = 2x_2 \qquad \begin{pmatrix} 2 \\ 1 \end{pmatrix} x_2$$

$$-2 - 2$$

$$P = \begin{pmatrix} -2 & 2 \\ 1 & 1 \end{pmatrix} \qquad \mathcal{D} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$$

$$P^{-1} = -\frac{1}{4} \begin{pmatrix} 1 & -2 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$$

3.

$$Z = P^{-1} Y = \begin{pmatrix} -\frac{1}{4}y_1 + \frac{1}{2}y_2 \\ \frac{1}{4}y_1 + \frac{1}{2}y_2 \end{pmatrix}$$

$$\begin{aligned} Z'(t) = P^{-1} Y'(t) &= P^{-1}(AY(t) + B(t)) \\ &= P^{-1} AP\, Z(t) + P^{-1}B(t) \\ &= \mathcal{D}\, Z(t) + P^{-1}B(t) \end{aligned}$$

Équation homogène associé

$$Z'(t) = \mathcal{D}\, Z(t)$$

$$\Leftrightarrow \begin{cases} z_1'(t) = -3\, z_1(t) \\ z_2'(t) = 5\, z_2(t) \end{cases} \Leftrightarrow \begin{cases} z_1(t) = C_1 e^{-3t} \\ z_2(t) = C_2 e^{5t} \end{cases}$$

$$\Leftrightarrow Z(t) = \begin{pmatrix} e^{-3t} & 0 \\ 0 & e^{5t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = e^{t\mathcal{D}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Equation complète :

Soit $Z \in C^1$ et $W(t) = e^{-tD} Z(t)$

$W'(t) = -De^{-tD} Z(t) + e^{-tD} Z'(t)$

$= -De^{-tD} Z(t) + e^{-tD}(D Z(t) + P^{-1}B(t))$

$= e^{-tD}P^{-1}B(t)$

donc $W'(t) = e^{-tD}P^{-1}B(t)$ et $W(t) = \int_0^t e^{-sD} P^{-1} B(s)ds + C$

Finalement, $Y(t) = PZ(t) = Pe^{tD}W(t)$

$= Pe^{tD}\int_0^t e^{-sD}P^{-1} B(s)ds + Pe^{tD}C$

$= P\int_0^t e^{(t-s)D} P^{-1} B(s)ds + Pe^{tD}C$

$e^{(t-s)D} = \begin{pmatrix} e^{-3(t-s)} & 0 \\ 0 & e^{5(t-s)} \end{pmatrix}$ $\begin{pmatrix} -1 & 2 \\ 1 & 2 \end{pmatrix}$

$e^{(t-s)D}P^{-1} = \frac{1}{4}\begin{pmatrix} -e^{-3(t-s)} & 2e^{-3(t-s)} \\ e^{5(t-s)} & 2e^{5(t-s)} \end{pmatrix}$

$e^{(t-s)D}P^{-1}B(t) = \frac{1}{4}\begin{pmatrix} -e^{-3(t-s)} & 2e^{-3(t-s)} \\ e^{5(t-s)} & 2e^{5(t-s)} \end{pmatrix}\begin{pmatrix} e^t \\ e^{-3t} \end{pmatrix}$

$= \frac{1}{4}\begin{pmatrix} -e^{-3(t-s)+t} + 2e^{-3(t-s)-3t} \\ e^{5(t-s)+t} + 2e^{5(t-s)-3t} \end{pmatrix}$

$= P\frac{1}{4}\int_0^t \begin{pmatrix} -e^{-3(t-s)+t} + 2e^{-3(t-s)-3t} \\ e^{5(t-s)+t} + 2e^{5(t-s)-3t} \end{pmatrix} ds + Pe^{tD}C$

$\int_0^t -e^{-3t+t+3s} ds$

$= -e^{-2t}\int_0^t e^{3s} ds = \left[\frac{-e^{-2t}}{3} e^{3s}\right]_0^t$

$= \frac{-e^t}{3} - \frac{e^{-2t}}{3}$

$2\int e^{-6t+3s} ds$

$= 2e^{-6t}\int_0^t e^{3s} ds$

$= \left[\frac{2}{3}e^{-6t}e^{3s}\right]_0^t$

$= \frac{2}{3}e^{-3t} + \frac{2}{3}e^{-6t}$

$\int_0^t e^{6t-5s} + 2e^{2t-5s} ds = \int_0^t e^{6t}e^{-5s} + 2e^{2t}e^{-5s} ds = \left[-\frac{e^{6t}}{5}e^{-5s} - \frac{2e^{2t}}{5}e^{-5s}\right]_0^t$

$= -\frac{1}{5}e^t - \frac{2}{5}e^{-3t} + \frac{e^{6t}}{5} + \frac{2}{5}e^{2t}$

$= P\frac{1}{4}\begin{pmatrix} \frac{-e^t}{3} - \frac{e^{-2t}}{3} + \frac{2}{3}e^{-3t} + \frac{2}{3}e^{-6t} \\ -\frac{1}{5}e^t - \frac{2}{5}e^{-3t} + \frac{e^{6t}}{5} + \frac{2}{5}e^{2t} \end{pmatrix} + Pe^{tD}C$

Exercice 2.6

$$y''' - 4y'' + 5y' - 2y = 0$$

$$Y = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} \qquad Y = \begin{pmatrix} y' \\ y'' \\ y''' \end{pmatrix} = \begin{pmatrix} 0y + 1y' + 0y'' \\ 0y + 0y' + 1y'' \\ 2y - 5y' + 4y'' \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Soit $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$ $\quad \chi_A(x) = \det(A - \lambda I) = \begin{vmatrix} -x & 1 & 0 \\ 0 & -x & 1 \\ 2 & -5 & 4-x \end{vmatrix} = -x \begin{vmatrix} -x & 1 \\ -5 & 4-x \end{vmatrix} + 2 \begin{vmatrix} 1 & 0 \\ -x & 1 \end{vmatrix}$

$$= -x(-x(4-x) + 5) + 2$$

$$= 4x^2 - x^3 - 5x + 2$$