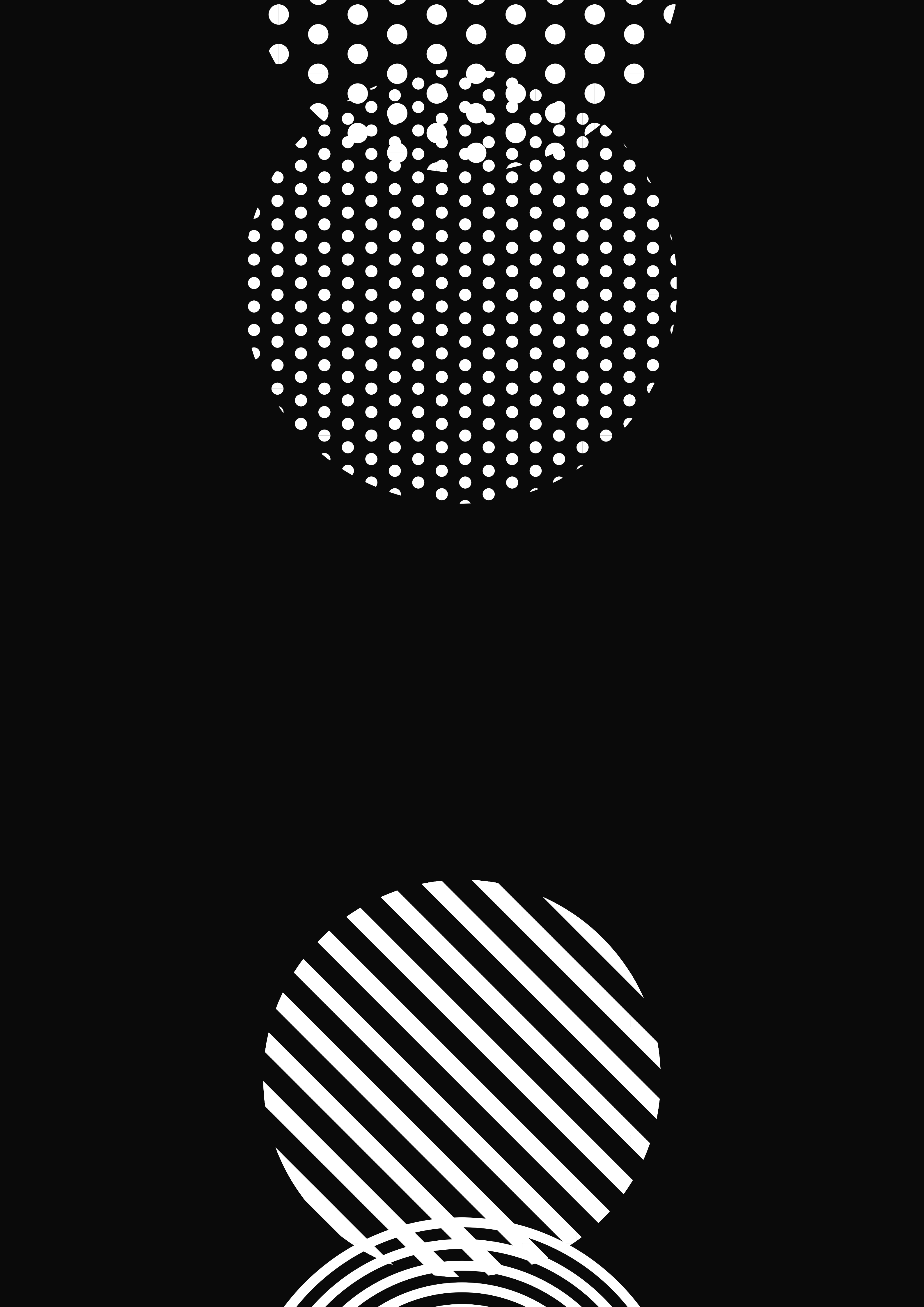

Exercice 2

1. $Y = X_3$

$$E[f(Y)] = \int_{\mathbb{R}} f(x^3)\frac{3}{\theta}x^2 e^{-\frac{x^3}{\theta}}dx = \int_{\mathbb{R}} f(y)\frac{1}{\theta} e^{-\frac{y}{\theta}} dy$$

$$y = x^3$$

$$dy = 3x^2$$

donc $Y \sim Exp(\frac{1}{\theta})$ donc $E[Y] = \theta$ et $Var(Y) = \theta^2$

2. $L(\theta) = \prod \frac{1}{\theta} e^{-\frac{y_i}{\theta}}$

$$= \frac{1}{\theta^N} e^{-\frac{1}{\theta}\sum y_i}$$

$$\ln L(\theta) = -N\log(\theta) - \frac{1}{\theta}\sum y_i$$

$$\vec{\nabla}\ln L(\theta) = -\frac{N}{\theta} + \frac{1}{\theta^2}\sum y_i \quad = 0 \quad \Leftrightarrow$$

$$\theta = \frac{\sum y_i}{N} = \overline{Y_N}$$

$\hat{\theta}_n = \overline{Y_i}$ par LGN c'est consistant

3. Par TCL $\sqrt{n}(\hat{\theta}_n - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, \theta^2)$

$$\Leftrightarrow \frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\theta} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$$

<u>Lemme de Slutsky</u>: $\hat{\theta}_n \xrightarrow{P} \theta > 0$, LAC $\frac{1}{\hat{\theta}_n} \xrightarrow{P} \frac{1}{\theta}$

On peut écrire $\frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\hat{\theta}_n} = \frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\theta} \cdot \frac{\theta}{\hat{\theta}_n} = \frac{\sqrt{n}(\hat{\theta}_n - \theta)}{\hat{\theta}_n} \xrightarrow{\mathcal{L}} \mathcal{N}(0,1)\cdot 1$

3 b)

On sait que

$$P\left(q_{\frac{\alpha}{2}} \le \frac{(\hat{\theta}_n - \theta)}{\sqrt{\hat{\theta}_n^2/n}} \le q_{1-\frac{\alpha}{2}}\right) \xrightarrow[n\to+\infty]{} 1-\alpha$$ où q_α est quantile d'ordre α.

$$q_{\frac{\alpha}{2}}\frac{\hat{\theta}_n}{\sqrt{n}} \le \hat{\theta}_n - \theta \le q_{1-\frac{\alpha}{2}}\frac{\hat{\theta}_n}{\sqrt{n}}$$

$$P\left(\hat{\theta}_n - q_{1-\frac{\alpha}{2}}\frac{\hat{\theta}_n}{\sqrt{n}} \le \theta \le \hat{\theta}_n - q_{\alpha/2}\frac{\hat{\theta}_n}{\sqrt{n}}\right) \xrightarrow[n\to+\infty]{} 1-\alpha$$

$$IC = \left[\hat{\theta}_n - q_{1-\frac{\alpha}{2}}\frac{\hat{\theta}_n}{\sqrt{n}}, \hat{\theta}_n - q_{\frac{\alpha}{2}}\frac{\hat{\theta}_n}{\sqrt{n}}\right]$$

4. D'après q° 3 $\quad \dfrac{\hat{\theta}_n - \theta}{\sqrt{\theta^2/n}} \xrightarrow{\mathcal{L}} \mathcal{N}(0, 1)$

$$q_{\frac{\alpha}{2}} \leq \frac{\hat{\theta}_n - \theta}{\sqrt{\frac{\theta^2}{n}}} \leq q_{1-\frac{\alpha}{2}} \iff \hat{\theta}_n - q_{1-\frac{\alpha}{2}} \frac{\theta}{\sqrt{n}} \leq \theta \leq \hat{\theta}_n - q_{\frac{\alpha}{2}} \frac{\theta}{\sqrt{n}}$$

$$\iff \hat{\theta}_n \leq \theta\left(1 - \frac{q_{1-\frac{\alpha}{2}}}{\sqrt{n}}\right) \text{ et } \hat{\theta}_n \geq \theta\left(1 - \frac{q_{\alpha/2}}{\sqrt{n}}\right)$$

$$\iff \frac{\hat{\theta}_n}{1 - \frac{q_{1-\frac{\alpha}{2}}}{\sqrt{n}}} \leq \theta \leq \frac{\hat{\theta}_n}{1 - \frac{q_{\alpha/2}}{\sqrt{n}}}$$