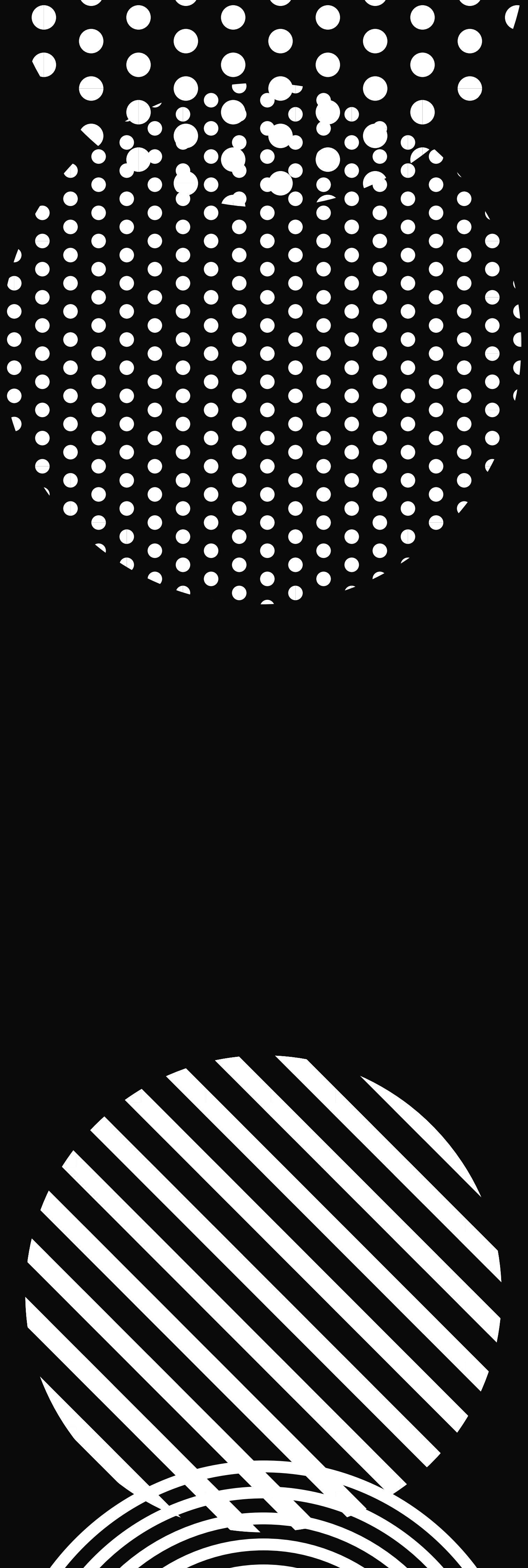

Exercice 1

1. $f_\theta(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{x^2}{2\theta}}$

V.S. : $L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = (2\pi\theta)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\theta}\sum_{i=1}^{n} X_i^2\right)$

log V.S : $\ell(\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum_{i=1}^{n} X_i^2$

dérivée : $\ell'(\theta) = -\frac{n}{2\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} X_i^2$

$= \frac{n}{2\theta^2}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \theta\right)$

On résout (par rapport à θ):

$\ell'(\theta) = 0$

$\Leftrightarrow \theta = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

On pose $T = \frac{1}{n}\sum_{i=1}^{n} X_i^2$

Vérifions qu'il s'agit d'un max globale:

$\ell'(\theta) = \frac{n}{2\theta^2} - \frac{1}{\theta^3}\sum_{i=1}^{n} X_i^2$

$\ell''(\theta) = \frac{n}{2T^2} - \frac{1}{T^3} nT = \frac{n}{2T^2} - \frac{n}{T^2} = -\frac{n}{2T^2} < 0$

Donc T est bien l'EMV de θ.

2. LGN : $\frac{1}{n}\sum X_i^2 \xrightarrow[n\to+\infty]{P} E[X_1^2] = \theta$

do

3. $\ell(\theta)^2 = \left(\frac{n}{2\theta^2}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \theta\right)\right)^2 = \frac{n^2}{4\theta^4}\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} X_i^2\right)^2 + \theta^2 - \frac{2}{n}\sum_{i=1}^{n} X_i^2\right)$

$E_\theta[\ell'(\theta)^2] =$

$I_n(\theta) = Var(S(\theta)) = Var(\ell'(\theta))$

$= Var\left(\frac{n}{2\theta^2}(T-\theta)\right)$

$= \left(\frac{n}{2\theta^2}\right)^2 Var(T)$

$= \left(\frac{n}{2\theta^2}\right)^2\left(\frac{1}{n^2}\sum_{i=1}^{n} Var(X_i^2)\right) = \frac{n^2}{4\theta^4}\cdot\left(\frac{1}{n^2}\sum E[X_i^4] - E[X_i^2]^2\right)$

$= \frac{1}{4\theta^2}\cdot\left(\frac{1}{n^2}\sum 3\theta^2 - \theta^2\right) = \frac{n^2}{4\theta^2}\cdot\frac{2\theta^2}{n} = \frac{n}{2\theta^2}$

La borne de Cramer-Rao pour tout estimateur sans-biais de θ si :

$$BCR(\theta) = \frac{1}{\mathcal{I}_n(\theta)} = \frac{2\theta^2}{n}$$

Or T est sans-biais et $Var(T) = BCR(\theta)$. Donc T est un estimateur efficace de θ.

Exercice 2

1. $$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i)$$

$$\log L(\theta) = \sum_{i=1}^{n} \log f_\theta(X_i)$$

$$= \sum_{i=1}^{n} \log\left(\frac{1}{\theta}\right) + \left(\frac{1}{\theta}-1\right)\log x_i$$

$$= \underbrace{n\log\left(\frac{1}{\theta}\right)}_{\searrow} + \underbrace{\left(\frac{1}{\theta}-1\right)}_{\searrow}\sum_{i=1}^{n}\log x_i \qquad \forall \theta > 0$$

$$\frac{\partial}{\partial\theta}\log L(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^{n}\log(x_i) \qquad \frac{\partial^2}{\partial\theta^2}\log L(\theta) = \frac{2n}{\theta^2} + \frac{2}{\theta^3}\sum\log(x_i)$$

$$\Leftrightarrow \frac{n}{\theta} = -\frac{1}{\theta^2}\sum_{i=1}^{n}\log(x_i)$$

$$\Leftrightarrow -\frac{n}{\sum\log(x_i)} = \frac{1}{\theta}$$

$$\Leftrightarrow \theta = -\frac{1}{n}\underbrace{\sum\log(x_i)}_{\leq 0} \geq 0$$

On pose $\hat{\theta}$, vérifions max glob.

$$L''(\theta) = \frac{n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n}\ln(x_i) = \frac{1}{\theta^3}\left(n\theta + 2\sum_{i=1}^{n}\ln(x_i)\right)$$

$$L''(\hat{\theta}) = \frac{1}{\hat{\theta}^3}\left(n\hat{\theta}^2 + 2\sum\ln(x_i)\right) = \underbrace{\frac{1}{\hat{\theta}^3}}_{\geq 0}\underbrace{\sum_{i=1}^{n}\ln(x_i)}_{\leq 0} \leq 0$$

$\hat{\theta}$ est bien l'EMV de θ.

2. $$Y_i = -\frac{1}{\theta}\log X_i$$

$$P\left(-\frac{1}{\theta}\log X_i < y\right)$$

$$P(\log X_i > -\theta y)$$

$$= P(X_i > e^{-\theta y})$$

$$= 1 - P(X_i \leq e^{-\theta y}) = 1 - \int_0^{e^{-\theta y}} \frac{1}{\theta} z^{\frac{1}{\theta}-1}\, dz \qquad z^{\frac{1}{\theta}}$$

$$= 1 - \left[z^{\frac{1}{\theta}}\right]_0^{e^{-\theta y}} = 1 - e^{-\theta y \frac{1}{\theta}} = 1 - e^{-y} = F_{Y_i}(y)$$

donc $Y_i \sim \mathcal{E}(1)$

Loi de Z. Les X_i sont i.i.d les Y_i sont aussi
donc $Z = \sum Y_i \sim \Gamma(n, 1)$

De plus $Z = -\frac{1}{\theta} \sum \ln(X_i) = \frac{n}{\theta} \hat{\theta} \iff \hat{\theta} = \frac{\theta}{n} Z$

Biais : $E[\hat{\theta}] = \frac{\theta}{n} E[Z] = \theta$

donc $\hat{\theta}$ est sans biais.

3. $Var(\hat{\theta}) = \left(\frac{\theta}{n}\right)^2 Var(Z) = \frac{\theta^2}{n}$

4. $l'(\theta) = -\frac{n}{\theta} - \frac{1}{\theta^2} \sum_{i=1}^{n} \ln(X_i) = \frac{n}{\theta^2}\left(-\theta - \frac{1}{n}\sum_{i=1}^{n} \ln X_i\right)$
$= \frac{n}{\theta^2}(\hat{\theta} - \theta)$

5.

$I_n(\theta) = Var(S(\theta)) = \left(\frac{n}{\theta^2}\right)^2 Var(\hat{\theta}) = \frac{n^2}{\theta^4} \frac{\theta^2}{n} = \frac{n}{\theta^2}$

$BCR = \frac{1}{I_n(\theta)} = \frac{\theta^2}{n} = Var(\hat{\theta})$ $\hat{\theta}$ est bien efficace.

Exercice 3

1) $E[e^{tX}] = \int_{-\infty}^{+\infty} e^{tx} f_\theta(x)\,dx$

$= \int_0^{+\infty} e^{tx} \theta^{-2} x e^{-\frac{x}{\theta}}\,dx$

$= \theta^{-2} \int_0^{+\infty} x \exp\left(-x\left(\frac{1}{\theta} - t\right)\right) dx$

On suppose $t < \frac{1}{\theta}$ sinon, l'intégrale diverge

$\Psi(t) = \theta^{-2}\left[\frac{-1}{\theta^{-1}-t} x \exp(-x(\theta^{-1}-t))\right]_0^{+\infty} + \frac{\theta^{-2}}{\theta^{-1}-t}\int_0^{+\infty} \exp(-x(\theta^{-1}-t))\,dx$

$= \frac{\theta^{-2}}{\theta^{-1}-t} \int_0^{+\infty} \exp(-x(\theta^{-1}-t))\,dx$

$= \frac{\theta^{-2}}{\theta^{-1}-t}\left[\frac{-1}{\theta^{-1}-t} \exp(-x(\theta^{-1}-t))\right]_0^{+\infty} = \frac{\theta^{-2}}{\theta^{-1}-t} \cdot \frac{1}{\theta^{-1}-t}$

$= \frac{\theta^{-2}}{(\theta^{-1}-t)^2} = \frac{\theta^{-2}}{\theta^{-2}(1-\theta t)^2}$

$= \frac{1}{(1-\theta t)^2}$

$\psi(t) = E[e^{tX_1}]$; $\psi'(t) = E[X_1 e^{tX_1}]$; $\psi''(t) = E[X_1^2 e^{tX_1}]$

$\psi'(0) = E[X_1]$; $\psi''(0) = E[X_1^2]$

Or, $\psi'(t) = 2\theta(1-\theta t)^{-3} \Rightarrow \psi'(0) = 2\theta$ $\psi''(t) = 6\theta^2(1-\theta t)^{-4}$
$= E[X_1] \Rightarrow \psi''(0) = 6\theta^2 = E[X_1^2]$