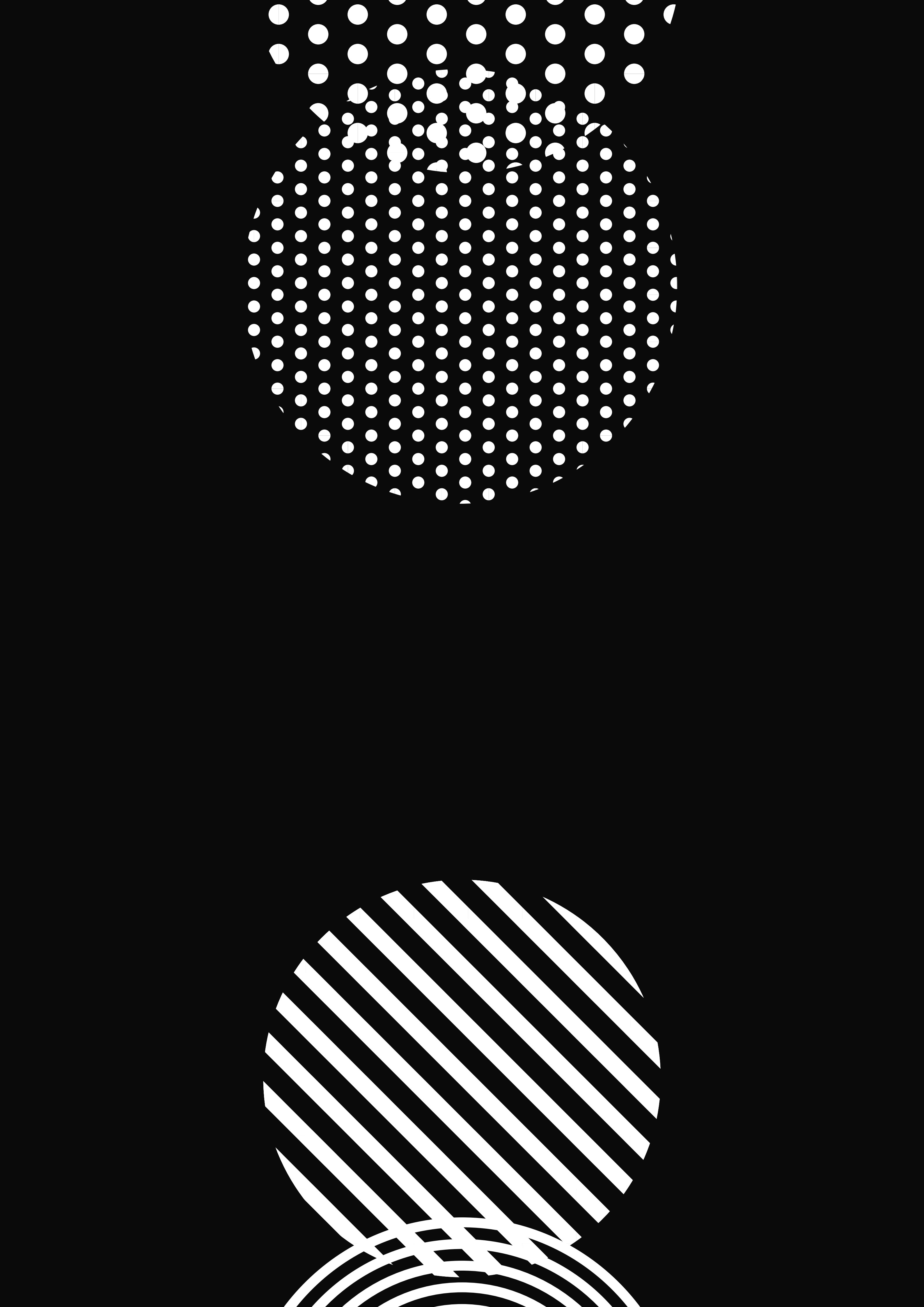

Exercice 1

1. f est ≥ 0

$$\int_{[\theta,+\infty[} e^{\theta-x}\,d\lambda = {}_{\theta}^{+\infty}[-e^{\theta-x}] = -e^{\theta-(+\infty)} + e^{\theta-\theta}$$
$$= -e^{-\infty} + e^{0}$$
$$= 0 + 1 = 1$$

$$P(X_i \leq x) = \int_{[\theta,x]} f(t)\,d\lambda(t)$$
$$= \Big[\underbrace{-e^{\theta-t}}\Big]_{\theta}^{x} = 1 - e^{\theta-x}$$
$$-e^{\theta-x}$$

2. $$E[X_i] = \int_{\theta}^{+\infty} x e^{\theta-x}\,dx = [-x e^{\theta-x}]_{\theta}^{+\infty} - \int_{\theta}^{+\infty} e^{\theta-x}\,dx$$
$$1 \qquad -e^{\theta-x} \qquad = \theta - [e^{\theta-x}]_{\theta}^{+\infty} = \theta + 1$$

$$E[X_i^2] = \int_{\theta}^{+\infty} x^2 e^{\theta-x}\,dx = [-x^2 e^{\theta-x}]_{\theta}^{+\infty} + 2\int_{\theta}^{+\infty} x e^{\theta-x}\,dx$$
$$2x \qquad = \theta^2 + 2E[X_i]$$
$$= \theta^2 + 2\theta + 2$$

$$Var(X_i) = E[X_i^2] - E[X_i]^2$$
$$= \theta^2 + 2\theta + 2 - \theta^2 - 2\theta - 1$$
$$= 1$$

2. ~~$E[\theta] = \frac{1}{n}\sum X_i \quad \overset{?}{=}$~~

On a vu dans la q° précédente $E[X_i] = \theta + 1$

La méthode des moments consiste à le paramètre d'intérêt θ en fonction du moment théorique $E[X_i]$

$$\theta = E[X_i] - 1$$

On remplace le moment théorique par son homologue empirique $\bar{X}_n$ pour obtenir l'estimateur des moments $\hat{\theta}$.

$$\hat{\theta} = \bar{X}_n - 1$$

$$B(\hat{\theta},\theta) = E[\hat{\theta}] - \hat{\theta} = E[\bar{X}_n - 1] - \hat{\theta}$$
$$= \frac{1}{n}\sum E[X_i] - 1 - \theta = \frac{1}{n}n(\theta+1) - 1 - \theta = 0$$

$$Var(\hat\theta_n) = Var(\bar X_n - 1) = Var(\bar X_n) = \frac{1}{n^2} Var(X_1 + \dots + X_n)$$

$$= \frac{Var(X_1)}{n} = \frac{1}{n}$$

Ainsi $\quad R(\hat\theta, \theta) = Var(\hat\theta) + B^2(\theta, \theta) = \frac{1}{n} \xrightarrow[n \to +\infty]{} 0$

donc cet estimateur est consistant.

3. a)

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f_\theta(X_i)$$

$$\ln \mathcal{L}(\theta) = \sum_{i=1}^{n} \ln f_\theta(X_i)$$

$$= \sum_{i=1}^{n} \ln e^{\theta - X_i}$$

$$= \sum_{i=1}^{n} \theta - X_i$$

$$= n\theta + \sum_{i=1}^{n} -X_i$$

$$\vec\nabla \ln \mathcal{L}(\theta) = \begin{pmatrix} n\theta - 1 \\ \vdots \\ n\theta - 1 \end{pmatrix}$$

$$\vec\nabla \ln \mathcal{L}(\theta) = 0 \iff n\theta - 1 = 0 \iff \theta = \frac{1}{n}$$

$$\cancel{\mathcal{L}(\theta) = \underset{\theta}{\arg\max}\ \mathcal{L}(\theta) = \min(X_1, \dots, X_n)}$$

Sol de prof :

$$L(\theta) = \prod_{i=1}^{n} f_\theta(x_i) = \prod_{i=1}^{n} \exp(\theta - x_i)\, \mathbb{1}_{[\theta, +\infty[}(x_i)$$

$$= \exp(n\theta - \sum_{i=1}^{n} x_i)\, \underline{\mathbb{1}_{[\theta, +\infty[}(\min x_i)}$$

Si $\theta > \min X_i$ $\quad L(\theta) = 0$, sinon > 0

Or pour $\theta \leq \min X_i$ $\quad \theta \mapsto \mathcal{L}(\theta)$ est st. croissante.

Ainsi, $\min X_i = \underset{\theta}{\arg\max}\ \mathcal{L}(\theta) = \hat\theta_{MV} = \min_i X_i$

b) Fct de répartition

Pour $z \geq 0$: $\quad P(\hat\theta_{MV} \leq z) = 1 - P(\hat\theta_{MV} > z) = 1 - P(X_1 > z, \dots, X_n > z)$

$$= 1 - P^n(X_1 > z)$$

$$= 1 - \exp(\theta - z)^n$$

$$= 1 - \exp(n(\theta - z))$$

Pour $z<0$, $P(\hat\theta_{MV} \le z) = 1 - P(\hat\theta_{MV} > z) = 0$

$$f_{\hat\theta_{MV}}(z) = n e^{n(\theta - z)} \mathbb{1}_{[\theta, +\infty[}(z)$$

c) $$E[\hat\theta_{MV}] = \int_\theta^{+\infty} \underset{u}{z}\, \underbrace{n e^{n(\theta-z)}}_{v'} dz \qquad u' = 1,\ v = -e^{n(\theta-z)}$$

$$= \left[-z e^{n(\theta-z)}\right]_\theta^{+\infty} + \int_\theta^{+\infty} e^{n(\theta-z)} dz$$

$$= \theta - \frac{1}{n}\left[e^{n(\theta-z)}\right]_\theta^{+\infty}$$

$$= \theta + \frac{1}{n}$$

$$B(\hat\theta_{MV}) = \frac{1}{n}$$

$$E[\hat\theta_{MV}^2] = \int_\theta^{+\infty} \underset{u}{z^2}\, \underset{v'}{n e^{n(\theta-z)}} dz \qquad u' = 2z,\ v = -e^{n(\theta-z)}$$

$$= \left[-z^2 e^{n(\theta-z)}\right]_\theta^{+\infty} + 2\int_\theta^{+\infty} z\, e^{n(\theta-z)} dz$$

$$= \theta^2 + \frac{2}{n} E[\hat\theta_{MV}]$$

$$= \theta^2 + \frac{2}{n}\left(\theta + \frac{1}{n}\right)$$

$$= \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2}$$

$$Var(\hat\theta_{MV}) = \cancel{\theta^2} + \cancel{\frac{2\theta}{n}} + \frac{2}{n^2} - \cancel{\theta^2} - \cancel{\frac{2\theta}{n}} - \frac{1}{n^2}$$

$$= \frac{1}{n^2}$$

$$R(\hat\theta_{MV}, \theta) = \frac{2}{n^2}$$

d) $E[\hat\theta_{MV}] = \theta - \frac{1}{n}$, on pose

$$\hat\theta_{MVC} = \hat\theta_{MC} - \frac{1}{n} = \min_i X_i - \frac{1}{n}$$

$$E[\hat\theta_{MVC}] = E[\hat\theta_{MV}] - \frac{1}{n} = \theta$$

Donc $\hat\theta_{MVC}$ est bien sans biais.

De plus $Var(\hat\theta_{MVC}) = Var(\hat\theta_{MV}) = R(\hat\theta_{MV}, \theta) - B^2(\hat\theta_{MV}, \theta)$
$$= \frac{2}{n^2} - \frac{1}{n} = \frac{1}{n^2}$$

$\hat\theta_{MVC}$ est consistant

4) Pour $n \geq 2$

$$R(\hat{\theta}_{MVC}, \theta) < R(\hat{\theta}_{MV}, \theta) < R(\tilde{\theta}, \theta)$$

$$\Leftrightarrow \quad \frac{1}{n^2} < \frac{2}{n^2} < \frac{1}{n}$$

Exercice 2

1) f dé: $f(-x) = f(x)$

$$E[X_1] = \int_{-\infty}^{0} x f(x) dx + \int_{0}^{+\infty} x f(x) dx$$

$$= 0$$

2) $E[X_1^2] = 2\theta^2 \quad \to \quad \theta = \sqrt{\frac{1}{2n} \sum X_i^2}$