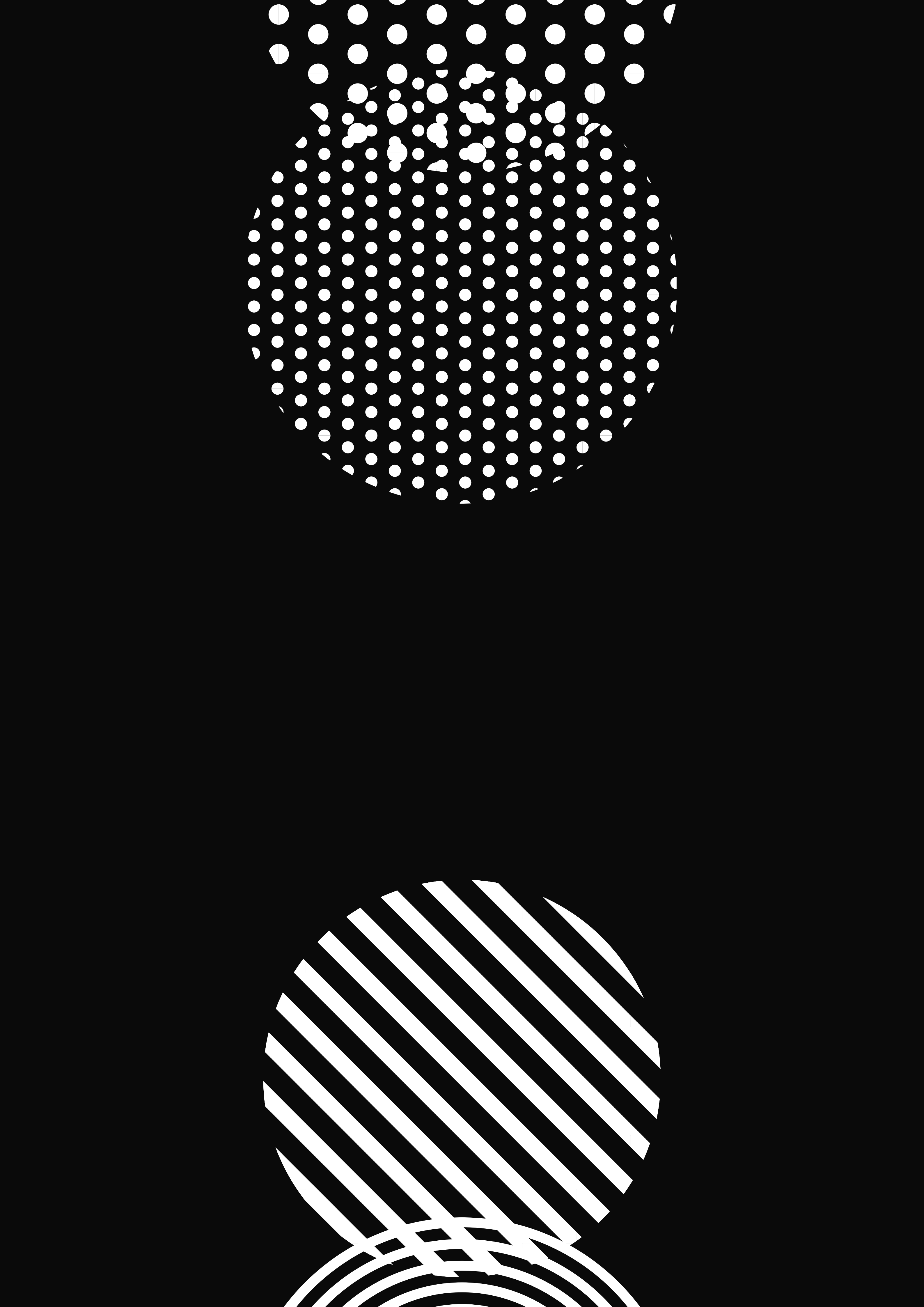
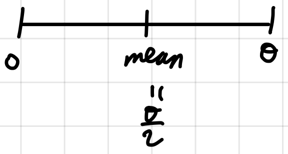


Exercice 1.

1.



donc $E[X_i] = \frac{\theta}{2} \iff \theta = 2E[X_i]$

$\downarrow$

méthode des moments

$$\hat{\theta}_M = 2\overline{X_n} = 2\frac{1}{n}\sum_{i=1}^{n} X_i$$

2. Max de vraisemblance = maximum likelihood

Chercher θ de façon à maximiser

$$p(X_1, \dots, X_n) = p(X_1) \dots p(X_n) = \prod_{i=1}^{n} f(X_i, \theta)$$

La densité d'une loi uniforme sur $[0, \theta]$

$$f(x, \theta) = \frac{1}{\theta} \mathbb{1}_{\{0 \leq x \leq \theta\}}$$

La fct de vraisemblance

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f(X_i, \theta) = \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}_{\{X_i \in [0,\theta]\}} = \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}_{\{X_i \in [0,\theta]\}}$$

$$= \frac{1}{\theta^n} \mathbb{1}_{\{0 \leq \max(X_1, \dots, X_n) \leq \theta\}}$$

On remarque que $\mathcal{L}(\theta) > 0 \iff \theta \geq \max(X_1, \dots, X_n)$

De plus $\theta \mapsto \frac{1}{\theta^n}$ Donc

$$\hat{\theta}_{MV} = \underset{\theta}{\arg\max}\ \mathcal{L}(\theta) = \max(X_1, \dots, X_n)$$

3. Biais : $E[\hat{\theta}_M] = 2E[\overline{X_n}] = 2E[X_i] = \theta$

$$B(\hat{\theta}_M, \theta) = E[\hat{\theta}_M] - \theta = 0$$

Risque quadratique : $R(\hat{\theta}_M, \theta) = Var(\hat{\theta}_M) + B(\hat{\theta}_M, \theta)$

$$= Var(\hat{\theta}_M)$$

$$= 4\,Var(\overline{X_n}) = \frac{4}{n^2}\sum_{i=1}^{n} Var(X_i) = \frac{4}{n} Var(X_i)$$

$$= \frac{\theta^2}{3n}$$

- Fct répartition

$$F_{\hat{\theta}_{MV}}(x) = P(\max(X_1, \dots, X_n) \leq x)$$
$$= P(X_1 \leq x)^n = \left(\frac{x}{\theta}\right)^n \mathbb{1}_{x \in [0,\theta]}$$

- Densité de probas $f_{\hat{\theta}_{MV}}(x) = F'_{\hat{\theta}_{MV}}(x) = \frac{n}{\theta^n} x^{n-1} \mathbb{1}_{x \in [0,\theta]}$
- Espérance $E[\hat{\theta}_{MV}] = \int_0^\theta x \cdot \frac{n}{\theta^n} x^{n-1} \, dx$

$$= \frac{n}{\theta^n} \int_0^\theta x^n \, dx = \frac{n\theta}{n+1}$$

- Biais $B(\hat{\theta}_{MV}, \theta) = E[\hat{\theta}_{MV}] - \theta = -\frac{\theta}{n+1}$

- Variance $E[\hat{\theta}_{MV}^2] = \int_0^\theta x^2 \frac{n}{\theta^n} x^{n-1} \, dx = \frac{n\theta^2}{n+2}$

$$Var(\hat{\theta}_{MV}) = E[\hat{\theta}_{MV}^2] - E[\hat{\theta}_{MV}]^2 = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2$$
$$= \frac{n\theta^2}{(n+1)(n+2)}$$

Risque quadratique

$$R(\hat{\theta}_{MV}, \theta) = \frac{n\theta^2}{(n+1)(n+2)} + \frac{\theta^2}{(n+1)^2} = \frac{2\theta^2}{(n+1)(n+2)}$$

Exercice 2

1. $E[X_i] = \frac{1}{p} \Rightarrow \hat{p}_M = \frac{1}{\bar{X}_n}$

2. $L(p) = \prod_{i=1}^n P(X_i = x_i) = \prod_{i=1}^n p(1-p)^{x_i - 1} = p^n (1-p)^{(\sum_{i=1}^n x_i - n)}$

pour maximiser $L(p)$ on maximise $\ell(p) = \ln(L(p))$

$$\ell(p) = n\ln(p) + \left(\sum_{i=1}^n x_i - n\right)\ln(1-p)$$

On dérive $\ell(p)$

$$\ell'(p) = \frac{n}{p} - \frac{\sum_{i=1}^n x_i - n}{1-p}$$

En posant $\ell(p) = 0$ $\quad \frac{n}{p} = \frac{\sum_{i=1}^n x_i - n}{1-p}$

$\Leftrightarrow$ $n(1-p) = p\left(\sum_{i=1}^n x_i - n\right)$

$\Leftrightarrow$

$$\Leftrightarrow n = p \sum x_i$$

$$\Leftrightarrow p = \frac{n}{\sum x_i} \Rightarrow \hat{p}_M = \frac{1}{\bar{X}_n}$$

On vérifie que $\hat{p}_M$ est une max. global

$$\ell''(p) = -\frac{n}{p^2} - \frac{\sum x_i - n}{(1-p)^2}$$

$$g = E[X] - 1 \quad \mu = 1$$

$$\frac{1}{x} = \frac{1}{2} = 2$$

$$E[\frac{1}{X}] = \sum_{i=1}^{n} \frac{1}{i} p(1-p)^{i-1}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \frac{1}{i} \left(\frac{1}{2}\right)^{i-1}$$

- $-\frac{n}{p^2} \leq 0$ car $n \in \mathbb{N}^*, p \in]0,1[$
- Le support d'une loi géométrique est $\mathbb{N}^*$. Donc $\forall i,\ x \geq 1$

$$\Rightarrow \sum x_i \geq \sum_{i=1}^{n} 1 = n \Rightarrow \sum_{i=1}^{n} x_i - n \geq 0$$

Ainsi, $\frac{\sum_{i=1}^{n} x_i - n}{(1-p)^2} \geq 0 \Leftrightarrow -\frac{\sum x_i - n}{(1-p)^2} \leq 0$

On en déduit $\ell''(p) < 0$ et donc ℓ concave, $\hat{p}_M$ est max global.

3. $\hat{p}_M = \hat{p}_{MM}$ par LGN, $\bar{X}_n = \frac{1}{p} > 0$

On applique le LAC à $x \mapsto \frac{1}{x}$ en $\frac{1}{p}$

On obtient $\hat{p}_{MM} = \frac{1}{\bar{X}} \xrightarrow[n \to +\infty]{p} \frac{1}{\frac{1}{p}} = p$

Exercice 3

1. $\mathcal{L}(\mu, \sigma^2) = \prod_{i=1}^{n} f(X_i, \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2\right)$

$$\ell(\mu, \sigma^2) = -\frac{n}{2} \ln(2\pi) - \frac{n}{2} \ln(\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2$$

Dérivée selon μ

$$\frac{\partial \ell}{\partial \mu} = \frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(X_i - \mu) = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \mu)$$

$$\frac{\partial \ell}{\partial \mu} = 0 \Leftrightarrow \sum_{i=1}^{n} (X_i - \mu) = 0$$

$$\Leftrightarrow \mu = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$= \bar{X}_n = \hat{\mu}$$

Dérivée selon σ^2

$$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \sum_{i=1}^{n} (x_i - \mu)^2 \left(-\frac{1}{2(\sigma^2)^2}\right) = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^{n} (x_i - \mu)^2}{2(\sigma^2)^2}$$

On cherche à annuler $\frac{\partial \ell}{\partial \sigma^2}$ en $\hat{\mu}$, càd

$$\left.\frac{\partial \ell}{\partial \sigma^2}\right|_{\hat{\mu} = \bar{X}_n} = 0 \iff -\frac{n}{2\sigma^2} + \frac{\sum (x_i - \hat{\mu})^2}{2(\sigma^2)^2} = 0$$

$$\iff n\sigma^2 = \sum_{i=1}^{n} (x_i - \hat{\mu})^2$$

$$\iff \sigma^2 = \frac{1}{n} \sum (X_i - \bar{X}_n)^2 = \hat{\theta}^2$$

Les équation n'admettent qu'une unique solution

$$(\hat{\mu}, \hat{\theta}^2) = \left(\bar{X}_n, \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X}_n)^2\right)$$

On veut vérifier que $(\hat{\mu}, \hat{\theta}^2)$ est un max global.
On calcul le Hessien en $(\hat{\mu}, \hat{\theta}^2)$