TD 3

Exercice 2

1. $E[X_i] = \frac{1}{2\theta}\int_{-\infty}^{+\infty} x e^{-\frac{|x|}{\theta}} = 0$ par symétrie.

2. $E[X_i^2] = 2\theta^2$

$\Leftrightarrow \theta^2 = \frac{E[X_i^2]}{2}$

$\Leftrightarrow \theta = \sqrt{\frac{E[X_i^2]}{2}}$

$\Leftrightarrow \hat{\theta} = \sqrt{\frac{\overline{X^2}}{2}}$

3. $\hat{\theta} - \theta = \sqrt{\frac{\sum x^2}{2n}} - \sqrt{\frac{E[X_i^2]}{2}}$

$= \sqrt{\frac{\frac{1}{n}\sum x^2 - E[X_i^2]}{2}}$

4. $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$

$= \prod_{i=1}^{n} \frac{1}{2\theta} e^{-\frac{|x_i|}{\theta}}$

$= \left(\frac{1}{2\theta}\right)^n e^{-\sum \frac{|x_i|}{\theta}}$

$\ln(\theta) = \ln\left(\frac{1}{2\theta}\right)^n - \sum \frac{|x_i|}{\theta}$

$= -n \ln 2\theta - \frac{1}{\theta}\sum |x_i|$

$= -n\ln 2\theta - \frac{1}{\theta}\sum |x_i|$

$= -n\ln 2 - n\ln\theta - \frac{1}{\theta}\sum |x_i|$

$\frac{\partial}{\partial\theta}\ln(\theta) = -n\frac{1}{\theta} + \frac{1}{\theta^2}\sum |x_i| = 0$

$\Leftrightarrow -n + \frac{1}{\theta}\sum |x_i| = 0$

$\Leftrightarrow \frac{1}{\theta}\sum |x_i| = n$

$\Leftrightarrow \sum |x_i| = n\theta$

$\Leftrightarrow \frac{\sum |x_i|}{n} = \hat{\theta}_{MV} = \overline{X}$

5. $B(\hat{\theta}_{MV}, \theta) = E[\hat{\theta}_{MV} - \theta] = E_{\hat{\theta}_{MV}}[X] - E[X] = 0$

$E[\overline{X}] = E[|X|]$

$P(|X| \in dx) = \frac{1}{\theta} e^{-\frac{|x|}{\theta}}$

$E[|X|] = \theta \qquad Var(|X|) = \theta^2$

$E[\hat{\theta}_{MV}] = \frac{1}{n}\sum E[|X_i|] = \theta$ donc $\hat{\theta}_{MV}$ est sans biais

$Var(\hat{\theta}_{MV}) = \frac{1}{n^2}\sum Var(|X_i|) = \frac{1}{n^2}\sum \theta^2 = \frac{\theta^2}{n}$

$$R(\hat{\theta}_{MV}, \theta) = \frac{\theta^2}{n} + 0 = \frac{\theta^2}{n} \xrightarrow[n \to +\infty]{} 0$$

6. Donc $\hat{\theta}_{MV} \to \theta$ en proba car $\hat{\theta}_{MV}$ est sans biais et le risque tend vers 0 lorsque $n \to +\infty$

Exercice 1

1. $$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{\theta}^{+\infty} e^{\theta - x}\,dx$$

$$= \left[-e^{\theta - x}\right]_{\theta}^{+\infty}$$

$$= -e^{-\infty} + e^{\theta - \theta}$$

$$= 0 + 1 = 1$$

De plus $\forall x \in \mathbb{R}$, $f(x) \geq 0$

$$P(X_1 \leq x) = \begin{cases} 0 & \text{si } x < \theta \\ \int_{\theta}^{x} e^{\theta - t}\,dt = \left[-e^{\theta - t}\right]_{\theta}^{x} \end{cases}$$

$$= -e^{\theta - x} + 1$$

$$= 1 - e^{\theta - x}$$

$$E[X_1] = \int_{\theta}^{+\infty} \underset{\substack{\downarrow \\ 1}}{x_u} \overset{\substack{\uparrow \\ -e^{\theta - x}}}{e^{\theta - x}_{v_1}}\,dx$$

$$= \left[-x e^{\theta - x}\right]_{\theta}^{+\infty} + \int_{\theta}^{+\infty} e^{\theta - x}$$

$$= \left[-x e^{\theta - x}\right]_{\theta}^{+\infty} + \left[-e^{\theta - x}\right]_{\theta}^{+\infty}$$

$$= \theta + 1$$

$$E[X_1^2] = \int_{\theta}^{+\infty} \underset{\substack{\downarrow \\ 2x}}{x^2} \overset{\substack{\uparrow \\ -e^{\theta - x}}}{e^{\theta - x}}\,dx$$

$$= \left[-x^2 e^{\theta - x}\right]_{\theta}^{+\infty} + 2\int_{\theta}^{+\infty} x e^{\theta - x}$$

$$= \theta^2 + 2E[X_1]$$

$$= \theta^2 + 2\theta + 2$$

$$Var(X_1) = E[X_1^2] - E[X_1]^2 = \theta^2 + 2\theta + 2 - (\theta^2 + 2\theta + 1)$$
$$= 1$$

2. $E[X_1] = \theta + 1 \iff \theta = E[X_1] - 1$ donc $\hat{\theta} = \frac{1}{n}\sum X_i - 1$

$$B(\hat{\theta}, \theta) = E[\hat{\theta}] - \theta = \frac{1}{n}\sum E[X_i] - 1 - \theta$$

$$= \frac{1}{n}\sum(\theta + 1) - 1 - \theta = \theta + 1 - 1 - \theta = 0$$

$$Var(\hat{\theta}) = \frac{1}{n^2}\sum Var(X_i) = \frac{1}{n^2}\sum 1 = \frac{1}{n}$$

3. (a) $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$

$$= \prod_{i=1}^{n} e^{\theta - x_i} \mathbb{1}_{[\theta,+\infty[}(x_i)$$

$$= \mathbb{1}_{[\theta,+\infty[}(\min x_i) \prod_{i=1}^{n} e^{\theta - x_i}$$

$$= \mathbb{1}_{[\theta,+\infty[}(\min x_i) \quad e^{n\theta - \sum x_i}$$

pour $\theta > \min x_i$ $\quad L(\theta) = 0$

pour $\theta \leq \min x_i$ $\quad L(\theta) = \exp(n\theta - \sum x_i)$

où $\theta \mapsto L(\theta)$ est strictement croissante.

Donc $\hat{\theta}_{MV} = \min_i x_i$

b) $P(\hat{\theta}_{MV} \leq x) = P(\min_i x_i \leq x) = 1 - P(\min_i x_i \geq x)$

$$= 1 - \prod_{i=1}^{n} P(x_i \geq x)$$

$$= 1 - e^{n(\theta - x)} = F(x)$$

$$\frac{dF}{dx}(x) = -n(-1)\, e^{n(\theta-x)} \underbrace{-e^{n(\theta-x)}}_{\nearrow} = n\, e^{n(\theta-x)} \mathbb{1}_{[\theta,+\infty[}(x)$$

(c) $E[\hat{\theta}_{MV}] = \int_\theta^{+\infty} \underset{\downarrow \atop 1}{x}\, \underbrace{n\, e^{n(\theta-x)}}\, dx = \left[-x\, e^{n(\theta-x)}\right]_\theta^{+\infty} + \int_\theta^{+\infty} e^{n(\theta-x)}\, dx$

$$= \theta + \left[-\frac{1}{n} e^{n(\theta-x)}\right]_\theta^{+\infty}$$

$$= \theta + \frac{1}{n}$$

$$E[\hat{\theta}_{MV}^2] = \int_\theta^{+\infty} \underset{\downarrow \atop 2x}{x^2}\, \underbrace{n\, e^{n(\theta-x)}}_{-e^{n(\theta-x)}} = \left[-x^2 e^{n(\theta-x)}\right]_\theta^{+\infty} + 2\int_\theta^{+\infty} x\, e^{n(\theta-x)}$$

$$= \theta^2 + 2\frac{1}{n}\left(\theta + \frac{1}{n}\right)$$

$$= \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2}$$

$$Var(\hat{\theta}_{MV}) = \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} - \theta^2 - \frac{2\theta}{n} - \frac{1}{n^2} = \frac{1}{n^2}$$

$$R(\hat{\theta}_{MV}, \theta) = Var(\hat{\theta}_{MV}) + B^2 = 2 - \frac{1}{n^2} + \frac{1}{n^2} = \frac{1}{n^2}$$

d)