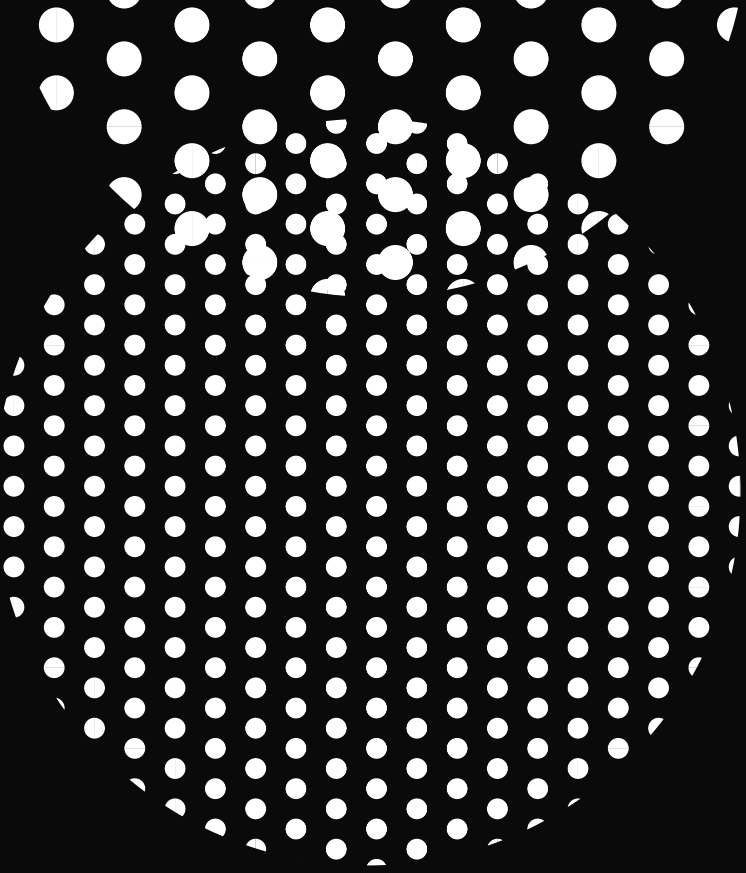

c)

$$\lim_{n\to+\infty} \langle f^{\varphi(n)}, u_k\rangle = f_k$$

$$\sum |f_k|^2 = \sum_{k=0}^{p} \lim_{n\to+\infty} \langle f^{\varphi(n)}, u_k\rangle \overline{\langle f^{\varphi(n)}, u_k\rangle}$$

$$= \lim_{n\to+\infty} \sum_{k=0}^{p} \langle f^{\varphi(n)}, u_k\rangle \overline{\langle f^{\varphi(n)}, u_k\rangle}$$

$$= \lim_{n\to+\infty} \langle \sum_{k=0}^{p} \langle f^{\varphi(n)}, u_k\rangle u_k , f^{\varphi(n)} \rangle$$

$$= \langle \sum_{k=0}^{n} f_k u_k , f_k \rangle$$

$$= \sum \langle$$

$$\sum_{k=0}^{p} |f_k|^2 = \sum_{k=0}^{p} |\lim_{n} \langle f^{\varphi(n)}, u_k\rangle|^2$$

$$= \lim_{n\to+\infty} \sum_{k=0}^{p} |\langle f^{\varphi(n)}, u_k\rangle|^2 \leq \lim_{n} \|f^{\varphi(n)}\|^2 \leq \lim_{n} 1 \leq 1$$

On pose $f := \sum_{k=0}^{+\infty} f_k u_k$

Puisque la borne $\sum_{k=0}^{p} |f_k|^2 \leq 1$ est uniforme en p, on obtient $(f_k)_k \in \ell^2_{\mathbb{K}}(\mathbb{N})$. Donc, par le cours, f est bien définie.
Toujours par le cours,

$$\langle f, u_k\rangle = \langle \sum_{n=0}^{+\infty} f_n u_n , u_k \rangle = f_k$$

d)

$$\| T(g - \pi_{F_p}(g)) \|$$

$$\sum_{k=0}^{p} \lambda_k \underbrace{\langle u_k^{\perp}, u_k\rangle}_{=0} u_k + \underbrace{\sum_{k=p+1}^{+\infty} \lambda_k \langle g, u_k\rangle u_k}_{\leq \sum_{k=p+1}^{+\infty} \lambda_k \cdot 1}$$

$$\leq M_p \sum_{k=p+1}^{+\infty} \langle g, u_k\rangle u_k$$

$$M_p = \sup_{k>p} |\lambda_k|$$

$$\pi_{F_p}(g) = \sum_{n=0}^{p} \langle g, u_n\rangle u_n$$

$$g - \pi_{F_p}(g) = \sum_{n=p+1}^{+\infty} \langle g, u_n\rangle u_n$$

$$T(g) = \sum_{k=0}^{+\infty} \lambda_k \langle g, u_k\rangle u_k$$

$$T(\pi_{F_p}(g)) = \sum_{k=0}^{+\infty} \lambda_k \langle \sum_{n=0}^{p} \langle g, u_n\rangle u_n , u_k\rangle u_k$$

$$= \sum_{k=0}^{+\infty} \lambda_k \left(\sum_{n=0}^{p} \langle g, u_n\rangle \langle u_n, u_k\rangle \right) u_k$$

$$= \sum_{k=0}^{p} \lambda_k \langle g, u_k\rangle u_k$$

$$T(g) - T(\pi_{F_p}(g)) = \sum_{k=p+1}^{+\infty} \lambda_k \langle g, u_k\rangle u_k$$

par linéarité de T : $\| T(g - \pi_{F_p}(g)) \| = \sum_{k=p+1}^{+\infty} |\lambda_k \langle g, u_k\rangle|^2$ $\overset{1.}{=}$

$$\leq \sup_{k>p} |\lambda_k|^2 \left(\sum_{k=p+1}^{+\infty} |\langle g, u_k\rangle|^2 \leq \|g\|^2 \sup_{k>p} |\lambda_k|^2\right.$$

Exercice 2

a) $\frac{d}{d\theta} T_n(\cos\theta) = \frac{d}{d\theta}\cos(n\theta) = \underline{-n\sin n\theta} = -\sin\theta\, T_n'(\cos\theta) = \frac{d}{d\theta} T_n(\cos\theta)$

$$\frac{d^2}{d\theta^2} T_n(\cos\theta) = \frac{d}{d\theta} -n\sin n\theta$$

$$= \underline{-n^2\cos n\theta} \qquad = -\cos\theta T_n'(\cos\theta) + (\sin\theta)^2 T_n''(\cos\theta)$$

$$n^2 T_n = x T_n' - (1-x^2) T''$$

D'où $(1-x^2) T_n'' - x T_n' = -n^2 T_n$

et donc $-n^2 T_n + n^2 T_n = 0$

b) $\phi_n'(x) = T''_n(x)\sqrt{1-x^2} + T'_n(x)\left(-x \frac{1}{\sqrt{1-x^2}}\right)$

$$= \frac{T''_n(x)(1-x^2) - T'_n(x)x}{\sqrt{1-x^2}} = \frac{-n^2 T_n(x)}{\sqrt{1-x^2}}$$

5] a) $\langle f, T_n \rangle = \int_{-1}^{1} \frac{f(x)T_n(x)}{\sqrt{1-x^2}} dx$

$$= \int_{-1}^{1} f(x) \frac{(-1)}{n^2} \phi_n'(x) dx$$

IPP $= -\frac{1}{n^2} \underbrace{\left[f(x)\, \phi_n(x) \right]_{-1}^{1}}_{=0} + \int_{-1}^{1} f'(x)\, \phi_n(x) dx$

$-1 < a < b < 1$

$$= \frac{1}{n^2}\int_{-1}^{1} f'(x)\,\phi_n(x) dx$$

$$= \frac{1}{n^2}\int_{-1}^{1} f'(x)\, T_n'(x) \sqrt{1-x^2}\, dx$$

$$= \frac{1}{n^2}\left[f'(x)\sqrt{1-x^2}\, T_n(x)\right]_{-1}^{1} - \frac{1}{n^2}\int \frac{f''(1-x^2) - x f'(x) T_n(x)}{\sqrt{1-x^2}} dx$$

$$= -\frac{1}{n^2}\int \frac{\left[f''(x)(1-x^2) - x f'(x)\right] T_n(x)}{\sqrt{1-x^2}} dx$$

$$\leq \frac{1}{n^2}$$

b) On veut mq $\min \int_{-1}^{1} |f(x) - P(x)|^2 \frac{d}{\sqrt{1-x^2}} \leq \frac{B}{n^3}$

$P_k = T_k\sqrt{\frac{2}{\pi}}$

$|\langle f, P_k\rangle|^2 = \frac{2}{\pi}\langle f, T_k\rangle^2 \leq \frac{2}{\pi}\left(\frac{A}{k^2}\right)^2 = \frac{2}{\pi}\frac{A^2}{k^4}$

$\|f - \pi_{\mathbb{R}_n[X]}(f)\|^2 \leq \frac{2A^2}{\pi}\sum_{k=n+1}^{+\infty}\frac{1}{k^4} \leq \frac{2A^2}{\pi}\int_n^{+\infty}\frac{1}{x^4}dx$

$$= \frac{2A^2}{\pi}\frac{1}{3n^3} = \frac{2A^2}{\pi}\cdot\frac{1}{n^3}$$

5) $\min \int_{1}^{1} |f(x) - P_n(x)|\, dx \leq \frac{C}{n^3}$

$\sqrt{1-x^2} \leq 1$, pour $x \in]-1, 1[$

$\frac{1}{\sqrt{1-x^2}} \geq 1$

$\forall g$ integrable $\int_{-1}^{1} |g|^2 dx \leq \int_{1}^{1} |g|^2 \frac{1}{\sqrt{1-x^2}} dx$

$\min \int_{-1}^{1} |f(x) - p(x)| dx \leq \int_{1}^{1} |f(x) - p(x)| \frac{d}{\sqrt{1-x^2}} \leq \frac{C}{n^3}$