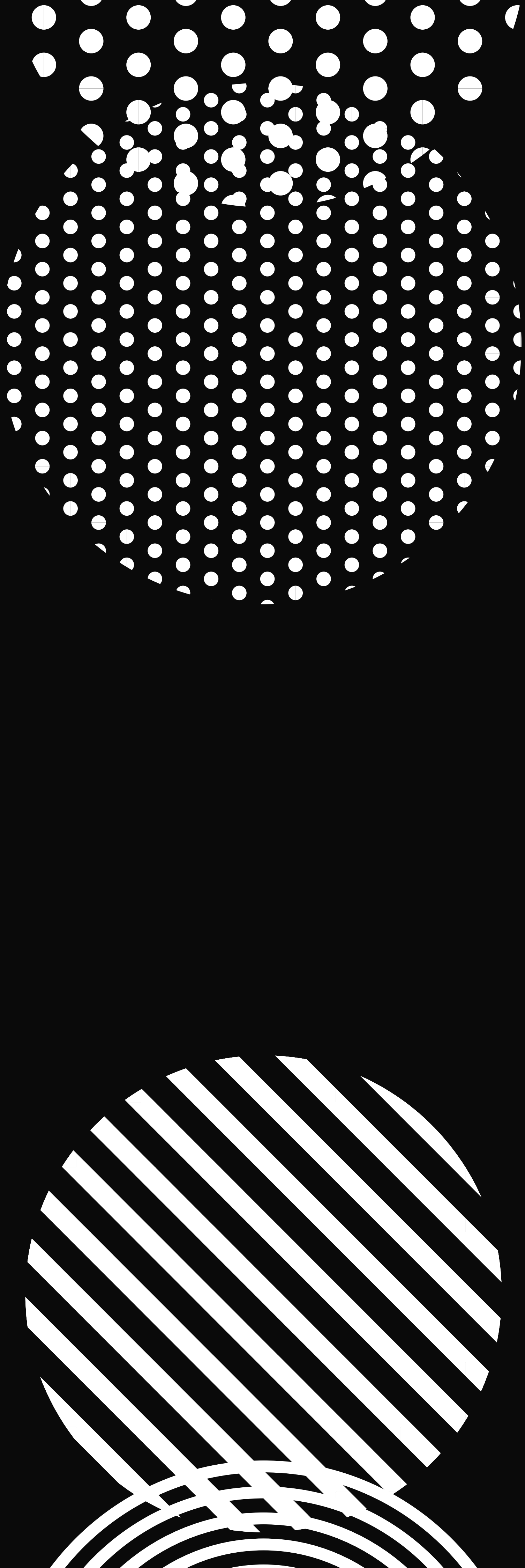


Exercice 1

1, 2, 3, 7, 8

$f(x) = \frac{1}{2} \langle x, Ax \rangle - \langle b, x \rangle$

$A = (a_{i,j})_{i,j}$

$f(x) = \frac{1}{2} \sum_i a_{i,j} x_i x_j$

$f(x+h) = f(x) + \langle \nabla f(x), h \rangle + o(h)$

ici : $f(x+h) = \frac{1}{2} \langle x+h, A(x+h) \rangle - \langle b, x+h \rangle$

$= f(x) + \frac{1}{2}\left(\langle x, Ah \rangle + \langle h, Ax \rangle \right) - \langle b, h \rangle$

$\nabla f(x) = \frac{1}{2}(A^T x + Ax) - b$

$= Ax - b$

A symétrique

$x_{n+1} = x_n - \alpha(Ax_n - b)$ (Richardson)

$\langle Ax, x \rangle = c \in \mathbb{R}$

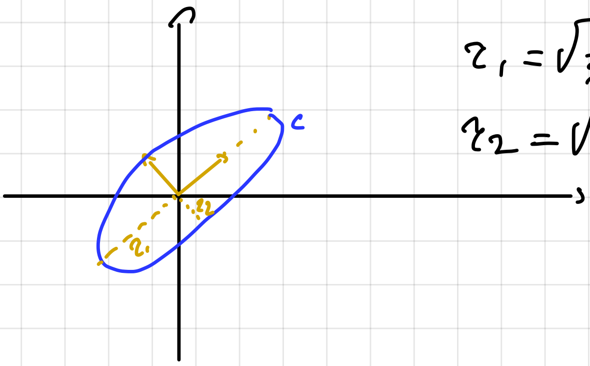
$A \in \mathcal{M}_2(\mathbb{R})$ SDP $\quad A \sim \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

u_1, u_2 base v.p (orthonormée)

$x = \langle x, u_1 \rangle u_1 + \langle x, u_2 \rangle u_2$

$Ax = \lambda_1 \langle x, u_1 \rangle u_1 + \lambda_2 \langle x, u_2 \rangle u_2$

$\langle Ax, x \rangle = \lambda_1 \langle x, u_1 \rangle^2 + \lambda_2 \langle x, u_2 \rangle^2$



$r_1 = \sqrt{\frac{c}{\lambda_1}}$

$r_2 = \sqrt{\frac{c}{\lambda_2}}$

$\frac{r_1}{r_2} = \sqrt{\frac{\lambda_2}{\lambda_1}}$

centre de l'ellipse $\nabla f(x) = 0$

$\iff Ax - b = 0$
$\iff x = A^{-1}b$

$\tilde{x} = x - A^{-1}b$

$f(x) = \frac{1}{2} < \tilde{x} + A^{-1}b, A(\tilde{x} + A^{-1}b)> - < b, \tilde{x} + A^{-1}b >$

$= \frac{1}{2}\left(<\tilde{x}, A\tilde{x}> + <\tilde{x}, b> + \underbrace{<A^{-1}b, A\tilde{x}>}_{<\tilde{x}, b>} + <A^{-1}b, b>\right) - <b, \tilde{x} + A^{-1}b>$

$f(\tilde{x} + A^{-1}b) = \frac{1}{2}<\tilde{x}, A\tilde{x}> - \frac{1}{2}<A^{-1}b, b>$

$\{x : f(x) = c\} = \{x + A^{-1}b : f(x + A^{-1}b) = c\}$
$= \{x + A^{-1}b : \frac{1}{2}<x, Ax> = c + \frac{1}{2}<A^{-1}b, b>\}$

$= A^{-1}b + \{x : \frac{1}{2}<x, Ax> = c + \frac{1}{2}<A^{-1}b, b>\}$
ellipses translatées
centrées en $A^{-1}b \in \mathbb{R}^n$

$x_0 = u_2$

$x_1 = u_2 - \lambda_2 \alpha u_2$

$= (1 - \lambda_2 \alpha) u_2$

$d(x_1, 0) = \| (1 - \lambda_2 \alpha) u_2 \|$

$= |1 - \lambda_2 \alpha|$

$\lambda_2 \alpha > 2 \implies d(x_1, 0) > 1$

Exercice 2

$f(x+h) = f(x) + < \nabla_M f(x), h >_M + o(h)$

$= f(x) + < \nabla f(x), h >_{eucl} + o(h)$

$= f(x) + < M M^{-1} \nabla f(x), h >_{eucl} + o(h)$

$= f(x) + < M^{-1} \nabla f(x), M h >_{eucl} + o(h)$

$\nabla_M f(x) = M^{-1} \nabla f(x)$

$= f(x) + < M^{-1} \nabla f(x), h >_M + o(h)$

Méthode de Newton trouver x^* tq $f(x^*) = 0$.

$x_{n+1} = x_n - \frac{f(x)}{f'(x)}$ $\qquad$ $x_{n+1} = x_n - Hess(f)(x_n)^{-1} \nabla f(x)$

$P\,Ax = P\,b$

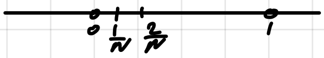
préconditionnement

$$x_{n+1} = x_n - \alpha(\nabla_u f(x_n))$$
$$= x_n - \alpha(\underbrace{\mathcal{M}^{-1}}_{=P} A x_n - \mathcal{M}^{-1} b)$$

Exercice 3

$$\begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$$

$x_k = \frac{k}{N}$



$$\mathcal{U} = (u(x_k))_{k=1,\dots,N-1}$$

$$u(x+h) = u(x) + h u'(x) + \frac{h^2}{2} u''(x) + \dots$$
$$u(x-h) = u(x) - h u'(x) + \frac{h^2}{2} u''(x) + \dots$$

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

$$u''(x_k) = \frac{u(x_{k+1}) - 2u(x_k) + u(x_{k-1})}{\frac{1}{N^2}} \qquad F\begin{pmatrix} f(x_1) \\ \vdots \\ f(x_{N-1}) \end{pmatrix}$$

$$\mathcal{M}\mathcal{U} = F$$

$$\mathcal{M} = N^2 \begin{pmatrix} 2 & -1 & & \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 2 \end{pmatrix} \qquad \text{taille } (N-1)\times(N-1)$$

$\mathcal{U}_0$ fixé $\qquad \mathcal{U}_{n+1} = \mathcal{U}_n - \alpha(\mathcal{M}\mathcal{U}_n - F) \qquad$ cv. ?

critère : $\mathcal{M}$ SDP $\qquad (\lambda_1 \leq \dots \leq \lambda_{N-1})$

Richardson cv ssi $\quad 0 < \alpha < \frac{2}{\lambda_{N-1}}$

$$\lambda_{N-1} = \max_{\|x\|=1} \langle \mathcal{M}x, x\rangle = \max_{\|x\|=1} N^2 \underbrace{\left(\sum_{i=1}^{N-1} x_i(2x_i - x_{i+1} - x_{i-1})\right)}_{4\|x\|^2} \leq 4N^2$$

$$e = \max(|1-\alpha\lambda_1|, |1-\alpha\lambda_{N-1}|) \qquad \mathcal{M}\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$$

$\lambda_{N-1} \propto N^2 \qquad \alpha = \frac{2}{\lambda_1 + \lambda_N} \sim \frac{c}{N^2} \qquad \|N\| \qquad \|\cdot\| = 2$

$\lambda_1 \propto 1$

$$\|\mathcal{U}_n - \mathcal{U}_*\| \leq e^n \sim \left(1 - \frac{c^2}{N^2}\right)^n \sim \exp\left(-n\frac{c^2}{N^2}\right) \leq \varepsilon \iff \frac{c^2 n}{N^2} \geq \log\left(\frac{1}{\varepsilon}\right) \iff n > N^2 \log\left(\frac{1}{\varepsilon}\right)$$