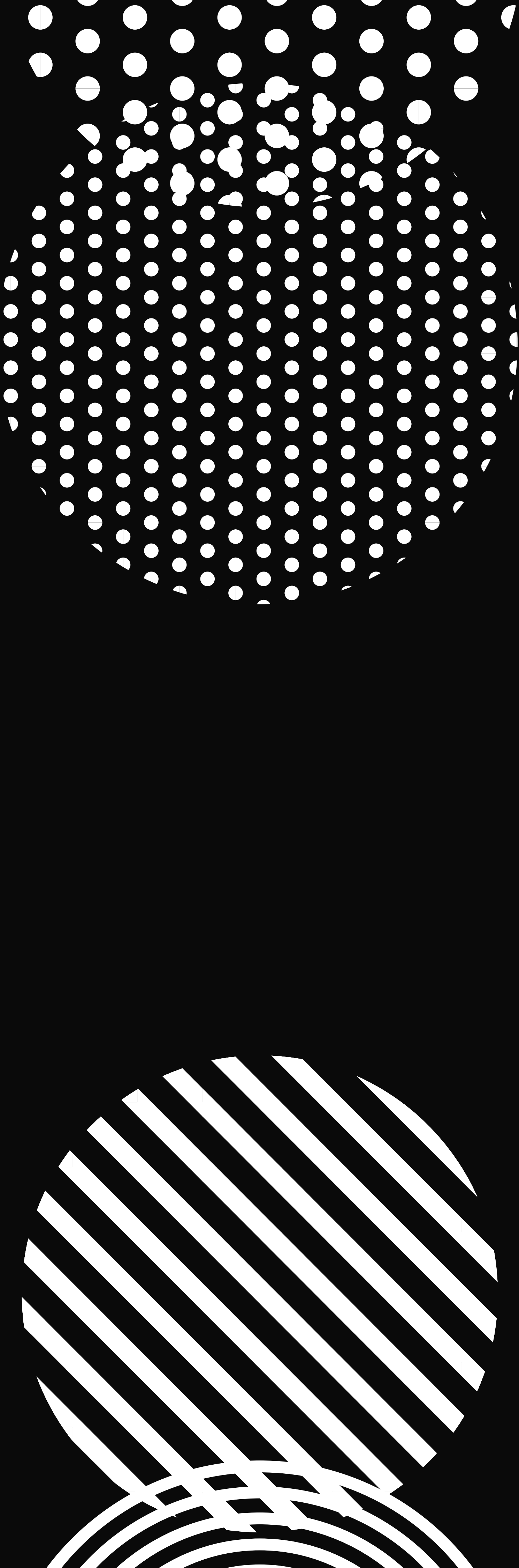


Exercice 1.

$$f(x+iy) = u(x,y) + i v(x,y)$$

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

$$\frac{\partial f}{\partial y} = \frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y}$$

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\partial \bar{z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\right)$$

$$\partial_{\bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x} u(x,y) + \frac{\partial}{\partial x} v(x,y) i + \frac{\partial}{\partial y} u(x,y) i - \frac{\partial}{\partial y} v(x,y)\right) = \frac{1}{2}\left((\partial_x u - \partial_y v) + i(\partial_x v + \partial_y u)\right) = 0$$

$$\Leftrightarrow \quad \partial_x u = \partial_y v \qquad \partial_x v = -\partial_y u$$

$$Jf = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$$

On écrit f comme DL :

$$f(z+h) = \overbrace{f(z)}^{\in \mathbb{C}} + \overbrace{f'(z)}^{\in \mathbb{C}} \overbrace{h}^{\in \mathbb{C}} + o(h)$$

$$f(z+h) = \underbrace{f(z)}_{\in \mathbb{R}^2} + \underbrace{df}_{\in \mathbb{R}^2} \cdot h + o(h)$$

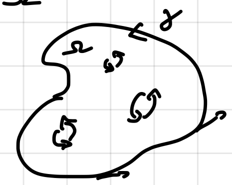
$$f'(z) = a + ib \; ; \quad f'(z)h = (a+ib)(h_x + i h_y)$$
$$= (a h_x - b h_y) + i(a h_y + b h_x)$$

$$df(z)\begin{pmatrix} h_x \\ h_y \end{pmatrix} = \begin{pmatrix} a h_x - b h_y \\ a h_y + b h_x \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} h_x \\ h_y \end{pmatrix}$$

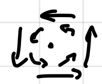
Exercice 2

Stokes

$$\iint_{\Omega} \mathrm{rot}(\vec{A})\, dx\, dy = \int_0^T \vec{A}(\gamma(s))\, \gamma'(s)\, ds$$



$\mathrm{rot} = 0$



$\mathrm{rot} \neq 0$

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \end{pmatrix} \qquad \mathrm{rot}\, \vec{A} = \begin{pmatrix} \partial x \\ \partial y \end{pmatrix} \wedge \begin{pmatrix} A_x \\ A_y \end{pmatrix} = \underset{\partial_x v}{\frac{\partial A_y}{\partial x}} - \underset{\partial_y u}{\frac{\partial A_x}{\partial y}}$$

Thm de Cauchy

Si f est holomorphe dans Ω, alors $\int_{\partial\Omega} f = 0$

$$\int_\gamma f = \int_0^T f(\gamma(s))\dot\gamma(s)ds = \int_0^T (u+iv)(\gamma(s))\dot\gamma(s)ds \overset{(*)}{=} \int_{\mathcal{C}} (u+iv)\left(d\vec{l}\cdot(\vec{e_x}+i\vec{e_y})\right)$$

$\gamma(s) = \begin{pmatrix}\gamma_x(s)\\ \gamma_y(s)\end{pmatrix}$ $\qquad \dot\gamma(s) = \dot\gamma_x(s) + i\dot\gamma_y(s)$

$$d\vec{l} = (\dot\gamma_x \vec{e_x} + \dot\gamma_y \vec{e_y})\,ds$$

$$= d\vec{l}(\vec{e_x} + i\vec{e_y})$$
$$= \dot\gamma_x\,ds + i\dot\gamma_y\,ds$$
$$= \dot\gamma(s)ds$$

$$(u+iv)(\vec{e_x}+i\vec{e_y}) = (u\vec{e_x} - v\vec{e_y}) + i(v\vec{e_x} + u\vec{e_y})$$

$$\int_{\partial\Omega = \mathcal{C}} f = (*) = \int_{\mathcal{C}} (u\vec{e_x} - v\vec{e_y})\,d\vec{l} + i\int (v\vec{e_x} + u\vec{e_x})\,d\vec{l}$$

$$\text{Stokes} = \iint_\Omega \underbrace{\left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)}_{=0} dxdy + i\iint_\Omega \underbrace{\left(-\frac{\partial v}{\partial y} + \frac{\partial u}{\partial x}\right)}_{=0} dxdy$$

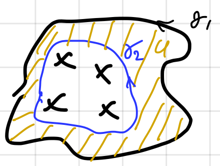
Cauchy-Riemann

$= 0$

Exercice 3

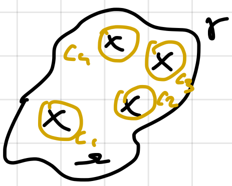
Déf: Un pôle de f est un point z_0 tel que $|f(z)| \xrightarrow[z\to z_0]{} +\infty$

Déf: f est méromorphe si $\forall z_0$ pôle de f, il existe $n \geq 1$ tel que $(z-z_0)^n f(z)$ est holomorphe près de z_0 } intuitive?



f est holomorphe dans U, donc $\int_{\partial U} f\,dz = 0$

donc $\int_{\gamma_1} f\,dz - \int_{\gamma_2} f\,dz = 0$ i.e. $\int_{\gamma_1} f\,dz = \int_{\gamma_2} f\,dz$



$$\oint f\,dz = \sum_i \int_{C_i} f\,dz$$

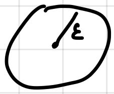
pôle en 0

$$\int_{\psi(0,\varepsilon)} f(z)dz = \int_{\psi(0,\varepsilon)} \frac{a_{n_0}}{z^{n_0}} + \dots + \frac{a_1}{z} + \underset{\substack{\shortparallel \\ 0 \text{ (partie holomorphe)}}}{g(z)}\,dz$$

Lemme: $\int_{\psi(0,\varepsilon)} \frac{1}{z^n}dz = \begin{cases} 2\pi i & \text{si } n = 1 \\ 0 & \text{sinon}\end{cases}$

$\gamma : [0, 2\pi] \longrightarrow \mathbb{C}$

$t \longmapsto \varepsilon e^{it}$



$$\int_{\mathcal{C}(0,\varepsilon)} f dz = \int_0^{2\pi} f(\gamma(t))\, \gamma'(t)\, dt$$

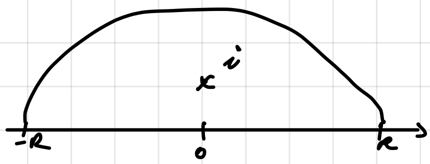
$$= \int_0^{2\pi} \frac{1}{(\varepsilon e^{it})^n} \varepsilon i e^{it}\, dt$$

$$= \int_0^{2\pi} e^{-i(n-1)t}\, dt \cdot i \varepsilon^{1-n}$$

$m \in \mathbb{Z}$ $$\int_0^{2\pi} e^{imt}\, dt = \begin{cases} 2\pi & \text{si } m = 0 \\ \left[\frac{e^{imt}}{im} \right]_0^{2\pi} = 0 & \text{si } m \neq 0 \end{cases}$$

Exercices

$\gamma(t) = \sqrt{R^2 - t^2}$ $$\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx$$



poles : $i, -i$

$$\int_{\mathcal{C}} \frac{1}{1+z^2}\, dz = \int_{-R}^{R} \frac{1}{1+\sqrt{R^2-t^2}} \frac{-2t}{\sqrt{R^2-t^2}}\, dt$$

$$\int_{\mathcal{C}_R} f(z)\, dz = 2\pi i \underbrace{\operatorname{Res}(i, f)}_{\frac{1}{2i}} = \pi$$

$$f(z) = \frac{a}{z-i} + g(z)$$

$$a = \lim_{z \to i} (z-i) f(z) = \lim_{z \to i} (z-i) \cdot \frac{1}{(z-i)(z+i)} = \frac{1}{2i}$$

$$\int_{\mathcal{C}_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx \Big] \longrightarrow \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\, dx$$

$$+ \left| \int f(z)\, dz \right|$$

$$= \left| \int_0^{\pi} f(\gamma(t))\, \gamma'(t)\, dt \right| = \left| \int_0^{\pi} \frac{1}{1+R^2 e^{it2}} R i e^{it}\, dt \right| \leq R \int_0^{\pi} \frac{1}{|1+R^2 e^{2it}|}\, dt$$

$$\leq R \int_0^{\pi} \frac{1}{R^2-1}\, dt$$

$$\leq \frac{R}{R^2-1} \pi \xrightarrow[R \to +\infty]{} 0$$