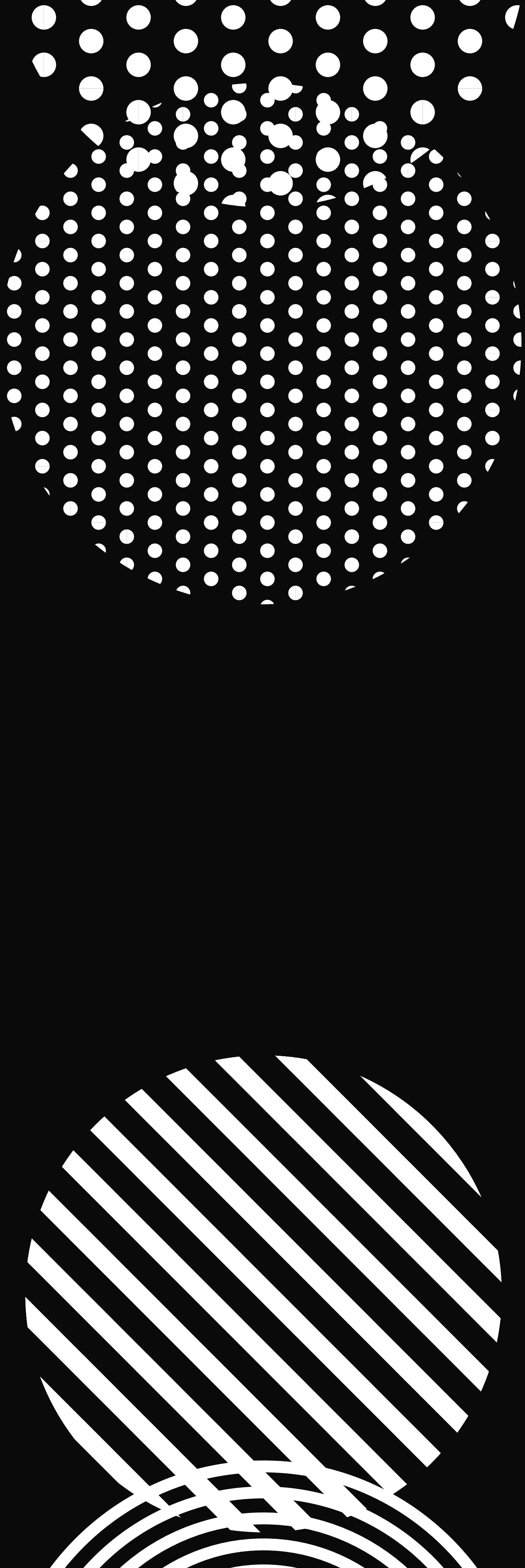


Exercice 4

a) A^*A

$$(A^*A)^* = A^*(A^*)^* = A^*A \quad : \text{symétrie}$$

définie positive: $\langle (A^*A)x, x\rangle > 0 \quad \forall x \in \mathbb{C}^N \setminus \{0\}$

Par définition, $\forall x \in \mathbb{C}^N,\ y \in \mathbb{C}^M \quad \langle Ay, x\rangle = \langle y, A^*x\rangle$

donc $\langle (A^*A)x, x\rangle = \langle Ax, Ax\rangle = \|Ax\|^2 \geq 0$

$\|Ax\|^2 = 0$ ssi $Ax = 0$ ssi $x = 0$ si A est injective (i.e les colonnes sont indépendantes)

b) Supposons $A^*A = LL^*$

$$(LL^*)^{-1} = (L^*)^{-1}L^{-1} = (A^*A)^{-1} = A^{-1}(A^*)^{-1}$$

On veut mq $\quad (A(L^*)^{-1})(A(L^*)^{-1})^* = I_d$

$$(A(L^*)^{-1})^* = ((L^*)^{-1})^* A^*$$

$$\begin{aligned}(A(L^*)^{-1})^* A(L^*)^{-1} &= ((L^*)^{-1})^* A^*A(L^*)^{-1}\\ &= ((L^*)^{-1})^* L \underbrace{L^*(L^*)^{-1}}_{I_d}\\ &= ((L^*)^{-1})^* L\\ &= L^{-1}L = I_d\end{aligned}$$

c) $$A = \underbrace{[A(L^*)^{-1}]}_{\substack{Q\\ \uparrow\\ \text{orth}}} \underbrace{L^*}_{R} \leftarrow \Delta.\ \text{sup}$$

$A = (v_1 \quad v_2)$

$$A^*A = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} (v_1 \quad v_2) = \begin{pmatrix} 1 & \langle v_1, v_2\rangle \\ \langle v_2, v_1\rangle & 1 \end{pmatrix}$$

$$\langle x, y\rangle = \sum_i \overline{x_i}\, y_i$$

$$(A^* A)_{i,j} = \sum_{k=1}^{n} (A^*)_{ik} A_{kj} = \sum_{k=1}^{n} \bar{A}_{ki} A_{kj} = \langle v_i, v_j\rangle$$

$$A = (v_1 | \dots | v_n)$$

$$L = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \qquad L L^* = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{b} \\ 0 & \bar{c} \end{pmatrix} = \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a} b & |b|^2 + |c|^2 \end{pmatrix}$$

$$a = \sqrt{\langle v_1, v_1\rangle} = \|v_1\| = 1$$

$$b = \langle v_2, v_1\rangle$$

$$c = \sqrt{1 - |\langle v_1, v_2\rangle|^2}$$

$$L = \begin{pmatrix} 1 & 0 \\ \langle v_2, v_1\rangle & \sqrt{1 - |\langle v_1, v_2\rangle|^2} \end{pmatrix}$$

Exercice 6

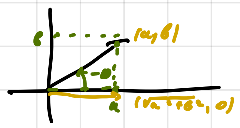
a)
$$G(i,j,\theta) = \begin{array}{c} \\ i \to \\ j \to \\ \\ \end{array} \begin{pmatrix} 1 & & & & \\ & \cos(\theta) & \cdots & -\sin(\theta) & \\ & \vdots & & \vdots & \\ & \sin(\theta) & \cdots & \cos(\theta) & \\ & & & & 1 \end{pmatrix}$$
$$\qquad\qquad\qquad\quad \uparrow_{i} \qquad\quad \uparrow_{j}$$

$$G(i,j,\theta)\begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ x_i' \\ \vdots \\ x_j' \\ \vdots \\ x_N \end{pmatrix} \qquad \begin{pmatrix} x_i' \\ x_j' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}\begin{pmatrix} x_i \\ x_j \end{pmatrix} \qquad z' = e^{i\theta} z$$

G est la rotation d'angle θ dans le plan $\mathrm{Vect}(e_i, e_j)$

b)
$$\cos(\theta) a - \sin(\theta) b = \sqrt{a^2 + b^2}$$

$$\sin(\theta) a + \cos(\theta) b = 0$$



$$\cos(-\theta) = \frac{a}{\sqrt{a^2+b^2}} = \cos(\theta)$$

$$\sin(\theta) = \frac{b}{\sqrt{a^2+b^2}} = \sin(-\theta) = -\sin\theta$$

$G^T G = Id$
$G^* G = Id$

$A = QR$

$Q^{-1} A = R$ $\qquad R = \begin{pmatrix} r_{1,1} & r_{1,2} & & \\ 0 & r_{2,1} & \ddots & \\ \vdots & 0 & & \ddots \\ 0 & \vdots & & \\ & 0 & & \end{pmatrix}$

$\exists \theta \quad G(1,2,\theta) A e_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix} = \begin{pmatrix} * \\ 0 \\ * \\ \vdots \end{pmatrix}$

Quand on effectue le produit $G(i,j,\theta)A$, on ne modifie que : a_{ii} ; a_{ji} ; a_{ij} ; a_{jj}

$G(1,n,\theta_n)\, G(1,n-1,\theta_{n-1}) \dots G(1,2,\theta_2) A = \begin{pmatrix} * & * & & * \\ 0 & * & \ddots & \\ \vdots & 0 & D & \\ 0 & 0 & & * \end{pmatrix}$ } WHY ?

G_1 et G_2 orthogonales :

$(G_1 G_2)^{-1} = G_2^{-1} G_1^{-1} = G_2^T G_1^T = (G_1 G_2)^T$

Exercice 10

- $f(x) = \|Ax - b\|^2 + \lambda \|x\|^2$

  f est continue et "coercive" ($f(x) \xrightarrow[\|x\| \to +\infty]{} +\infty$)
  donc admet un minimum global (⚠ pas forcément unique)

- Soit x un minimum de f. Alors $\nabla f(x) = \vec{0}$

  $f(x+h) = f(x) + \langle \nabla f(x), h \rangle + o(h)$

  $f(x+h) = \langle A(x+h) - b, A(x+h) - b \rangle + \lambda \langle x+h, x+h \rangle$

  $= f(x) + \underbrace{2\langle A^*(Ax-b) + \lambda x}_{\nabla f(x)}, h \rangle$

  $\vec{\nabla} f(x) = 0 \Leftrightarrow (A^*A + \lambda I_n) x = A^* b$

  $(A^*A + \lambda I_n) x = 0$

  Comme $\mathrm{Spec}(A^*A) \subset \mathbb{R}_+$ on a

  $\Rightarrow \langle (A^*A + \lambda I) x, x \rangle = 0 \Rightarrow \|Ax\|^2 + \lambda \|x\|^2 = 0 \Rightarrow \underline{x = 0}$