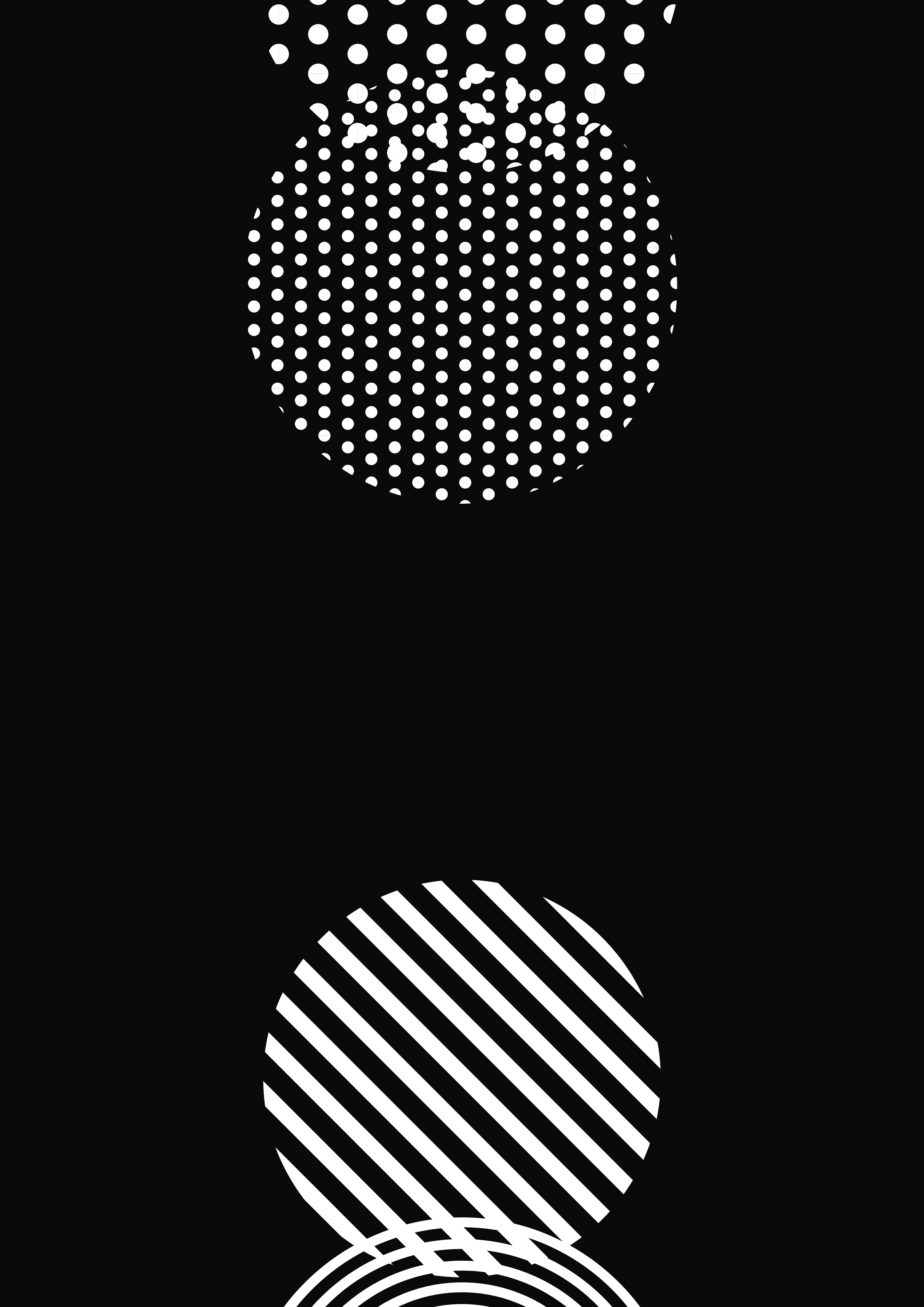
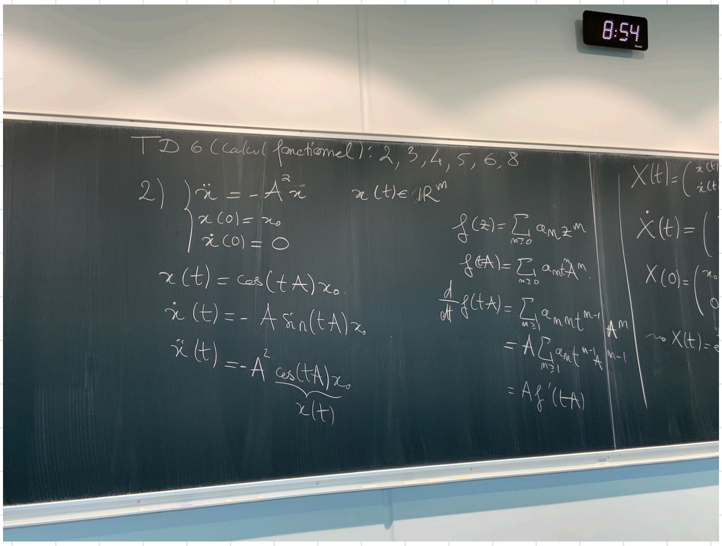
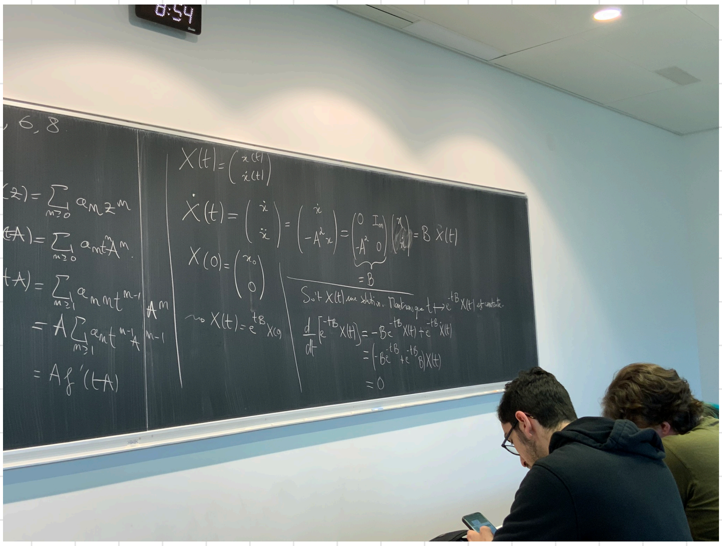


Exercice 2





Exercice 3

$A^* = -A$

$$\exp(A)^* = \Big(\sum_{n\geq 0} \frac{1}{n!} A^n\Big)^* = \sum_{n\geq 0} \frac{1}{n!}(A^*)^n$$

$$= \sum_{n\geq 0} \frac{(-1)^n}{n!} A^n = \exp(-A) \quad = \exp(A)^{-1}$$

Exercice 4

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$\chi_J = x^2 + 1$ donc $Sp(J) = \{-i, i\}$

$$J\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \Leftrightarrow \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} = i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$\Leftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix}$ puis $\begin{pmatrix} -i \\ 1 \end{pmatrix}$ pour $-i$

$$P^{-1} = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$P = \frac{1}{2i}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = -\frac{i}{2}\begin{pmatrix} 1 & i \\ -1 & i \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}$$

$$\exp(tJ) = \exp(t P^{-1} \mathcal{D} P)$$

$$= P^{-1}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix} P = \frac{1}{2}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} ie^{it} & -ie^{-it} \\ e^{it} & e^{-it} \end{pmatrix}\begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e^{it} + e^{-it} & i(e^{it} - e^{-it}) \\ -i(e^{it} - e^{-it}) & e^{it} + e^{-it} \end{pmatrix}$$

$$= \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

$$J^2 = -Id$$

$$e^{tJ} = \sum_{n\geq 0} \frac{t^{2n}}{(2n)!} \underbrace{J^{2n}}_{(-1)^n I_2} + \sum_{n\geq 0} \frac{t^{2n+1}}{(2n+1)!} \underbrace{J^{2n+1}}_{(-1)^n J}$$

$$= \cos(t) I_2 - \sin(t) J = \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

$$\frac{d^2}{dt^2} e^{tJ} = -e^{tJ}$$

$$x''(t) = -x(t)$$

Exercice 5

$$\exp(A+B) = \sum_{n\geq 0} \frac{1}{n!} (A+B)^n = \sum_{n\geq 0} \frac{1}{n!} (A+B)^{n-1} (A+B)$$

$$\exp\left(\frac{A}{n}\right) \exp\left(\frac{B}{n}\right) = \left(\sum_{n\geq 0} \frac{1}{n!} \frac{1}{n^n} A^n\right)\left(\sum_{n\geq 0} \frac{1}{n!} \frac{1}{n^n} B^n\right)$$

mq $\left(Id + \frac{A}{n}\right)^n \xrightarrow[n\to+\infty]{} e^A$

$$\cdot \left(1 + \frac{x}{n}\right)^n = e^{n\log\left(1+\frac{x}{n}\right)}$$

$$= e^{n\left(\frac{x}{n} + o(1)\right)}$$

$$= e^{x+o(1)}$$

$$\cdot \left(Id + \frac{A}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} A^k$$

$$= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \frac{1}{n^k} A^k$$

$$= \left(\sum_{k=0}^{n} \frac{1}{k!} \underbrace{\frac{(n-k)!(n-k+1)\dots n}{n\dots n\ (n-k)!}}_{\xrightarrow[n\to+\infty]{} 1 \text{ pour } k \text{ fixé}} A^k\right) \approx \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$$

interchanger limites ?

thm de cv. dominée

$$\sum_{k\geq 0} u_{n,k} A^k \longrightarrow \sum_{k\geq 0} \frac{1}{k!} A^k$$

$\| u_{n,k} A^k \| \leq u_k \in \mathbb{R}_+$ tq $\sum u_k < +\infty$

$\|A^k\| \leq \|A\|^k$ $\qquad a_{n,k} = \frac{1}{k!} \frac{n(n-1)\dots(n-k+1)}{n^k} \leq \frac{1}{k!}$

on met $u_k = \frac{1}{k!} \|A\|^k$ où $\sum u_k = \exp(\|A\|)$

donc $\exp(A) = \lim \left(Id + \frac{A}{n}\right)^n$

$$\exp(A+B) = \lim \left(Id + \frac{A+B}{n}\right)^n$$

$$= \lim \left(Id + \frac{A}{n} + \frac{B}{n}\right)^n$$

$$\left(\exp\left(\frac{A}{n}\right)\exp\left(\frac{B}{n}\right)\right)^n = \left(\left(Id + \frac{A}{n} + \frac{R_1^{(n)}}{n^2}\right)\left(Id + \frac{B}{n} + \frac{R_2^{(n)}}{n^2}\right)\right)^n$$

$$= \left(\left(Id + \frac{A+B}{n} + \frac{R_3^{(n)}}{n^2}\right)\right)^n \quad \text{avec} \quad \|R_3^{(n)}\| \leq C \text{ indep. de } n.$$

$$= \sum_{k \geq 0} \binom{n}{k} \frac{1}{n^k} \frac{n(A+B) + R_3^{(n)}}{n}$$

$$\longrightarrow \sum_{k \geq 0} \frac{1}{k!} \left((A+B) + \underbrace{\frac{R_3^{(n)}}{n}}_{\xrightarrow{n \to +\infty} 0}\right)^k$$

$$\left\|\left(A + \frac{R_3^{(n)}}{n}\right)^k\right\| \leq \left\|A + \frac{R_3^{(n)}}{n}\right\|^k \leq (\|A\| + C)^k$$

$$= \sum_{k \geq 0} \frac{1}{k!} (A+B)^k$$

$$= \exp(A+B)$$