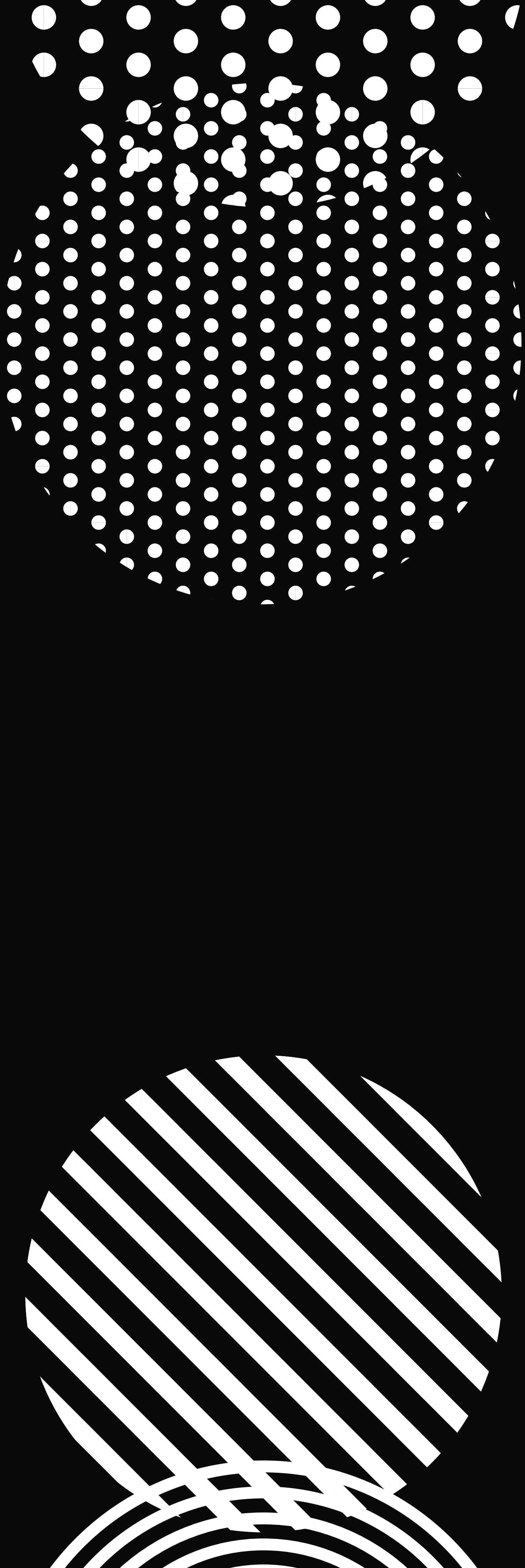

Excercice 1

$A \in \mathbb{C}^{N\times M}$

$\{\text{vals sing.}\} = \{\sqrt{\lambda_j} : \lambda_j \in \operatorname{spec}(A^*A)\}$

Soient U, V unitaires (orthogonales) tq $A = USV^*$

Soit T unitaire de taille $N\times N$

$TA = \underbrace{TU}_{\text{orthogonale}} SV^*$ (si T' est unitaire de taille $M\times M$, $AT' = US(V^*T')$)

• $(TA)^* TA = A T^* TA = A^*A$

$\Rightarrow \operatorname{Spec}((TA)^* TA) = \operatorname{Spec}(A^*A)$

$(AT')^* AT' = T'^* A^* AT' = (T')^{-1} A^* AT'$

Or $\operatorname{spec}(P^{-1}BP) = \operatorname{Spec}(B)$

Donc $\operatorname{Spec}((AT')^* AT') = \operatorname{Spec}(A^*A)$

Exercice 2

$B \in \mathbb{C}^{N\times N}$ HDP

$\lambda_1(B) \leq \dots \leq \lambda_N(B)$

$\lambda_1(B) = \min_{\|x\|=1} \langle Bx, x\rangle$

$\lambda_N(B) = \max_{\|x\|=1} \langle Bx, x\rangle$

$C \in \mathbb{C}^{N\times 1}$

$A' = (A|C)$

On note $s_1(A) \leq \dots \leq s_N(A)$ les valeurs singulières de A

$s_1(A)^2 = \lambda_1(A^*A) = \min_{\|x\|=1} \langle A^*Ax, x\rangle = \min_{\|x\|=1} \|Ax\|^2$

$s_1(A')^2 = \min_{\|y\|=1} \|A'y\|^2$

Soit $x \in \mathbb{C}^M$, $x' = \begin{pmatrix} x \\ 0 \end{pmatrix}$ $\left(x = \sum_{i=1}^{M} x_i e_i, \ x' = \sum_{i=1}^{M+1} x_i' e_i\right.$

$A'x' = Ax$

$A'x' = \underbrace{\sum_{i=1}^{M} x_i' A_i}_{Ax} + \underbrace{x'_{M+1} C}_{=0}$

$$\|A'x'\| \le \|Ax\| \quad \Rightarrow \quad \min_{\|x\|=1} \|A'x'\|^2 \le \min_{\|x\|=1} \|Ax\|^2$$

$$\Rightarrow \quad s_1(A')^2 \le s_1(A)^2$$

Exercice 3

$$A = \begin{pmatrix} \frac{1}{100} & 0 \\ 0 & 100 \end{pmatrix}$$

$\mathrm{Spec}(A) = \{ \frac{1}{100}, 100 \}$

Spec sing $= \{ \frac{1}{100}, 100 \}$ car $A^* = A$ et $\sqrt{A^*A} = \sqrt{A^2} = A$

$\det(A) = 1$

$$\kappa(A) = \|A\| \cdot \|A^{-1}\| = \frac{s_2(A)}{s_1(A)} = 10000$$

Exercice 4

$A \in \mathbb{C}^{n \times 2}$ tq

$$A^*A = \begin{pmatrix} 1 & a \\ a & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & a \\ a & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - a^2$$

$$\Leftrightarrow \begin{cases} 1-\lambda = a \\ \lambda - 1 = a \end{cases} \Leftrightarrow \begin{array}{l} \lambda = 1-a \\ \lambda = 1+a \end{array}$$

Sings $= \{ \sqrt{1-a}, \sqrt{1+a} \}$

Exercice 5

Si $A = \mathcal{U} S V^*$ $\quad B = \mathcal{U} V^*$ $\quad$ mq $B = A(A^*A)^{-\frac{1}{2}}$

$A^*A = VS^2V^*$

Donc $(A^*A)^{-\frac{1}{2}} = \left((VS^2V^*)^{-1}\right)^{\frac{1}{2}} = (VS^{-2}V^*)^{\frac{1}{2}} = VS^{-1}V^*$

$A(A^*A)^{-\frac{1}{2}} = USV^*VS^{-1}V^* = UV^*$

$A - B = A - A(A^*A)^{-\frac{1}{2}} = A(\mathrm{Id} - (A^*A)^{-\frac{1}{2}})$

$A - B = USV^* - UV^* = V(S - \mathrm{Id})V^*$

$\|C\|_F = \sqrt{\mathrm{Tr}(C^*C)}$

On a $\|A - B\|_{op} = \|S - \mathrm{Id}\|_{op} = \max\{|s_j(A) - 1| : s_j(A)$ val. sing$\}$

$\|A - B\|_F = \underset{\uparrow \text{ diagonale}}{\|S - \mathrm{Id}\|_F} = \sqrt{\sum_{1 \le j \le N} (s_j(A) - 1)^2}$

$A = QR$

$\|A - Q\|_{op} = \|Q(R - \mathrm{Id})\|_{op} = \|R - \mathrm{Id}\|_{op}$

$\|A - Q\|_F = \|R - \mathrm{Id}\|_F$

Exercice 7

$A^*A = VS^2V^*$ $\quad A = USV^*$

$(SV^*)^* = VS$

$(VSSV^*)^* = VSSV^*$

$A^*A = H^2$ $\quad \Rightarrow H = $ "$\sqrt{A^*A}$" $\quad Q = A(A^*A)^{-\frac{1}{2}}$ $\quad Q := UV^*$

donc $A = \underbrace{UV^*}_{Q} \underbrace{VSV^*}_{R}$

Exercice 12

On note $A = Id$ $\quad B = \begin{pmatrix} 0 & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix}$ $\quad B^2 = \begin{pmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon^2 \end{pmatrix}$

$\|B\| = \varepsilon$ $\quad \|A\| = \|A^{-1}\| = 1$

d'où $\|B\| \leq \frac{1}{\|A^{-1}\|} \leq 1 \iff |\varepsilon| < 1$

d'où $A - B$ est inversible et $(A-B)^{-1} = A^{-1} + A^{-1}BA^{-1} + A^{-1}BA^{-1}BA^{-1}$

$$= Id + B + B^2 + B^3 + \dots$$

$$= \begin{pmatrix} 1 + \sum_{k=1}^{+\infty} \varepsilon^{2k} & -\sum_{k=0}^{N} \varepsilon^{2k+1} \\ -\sum_{k=0}^{N} \varepsilon^{2k+1} & 1 + \sum_{k=1}^{+\infty} \varepsilon^{2k} \end{pmatrix}$$

$$(A-B)^{-1} = \sum_{k=0}^{+\infty} \varepsilon^{2k} I - \sum_{k=0}^{+\infty} \varepsilon^{2k+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \frac{1}{1-\varepsilon^2} I - \frac{\varepsilon}{1-\varepsilon^2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{1}{1-\varepsilon^2} \begin{pmatrix} 1 & -\varepsilon \\ -\varepsilon & 1 \end{pmatrix}$$

Rq: À l'ordre 1 $\quad (A-B)^{-1} = I_2 - \varepsilon \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

$\|A^{-1}\|_{op} = 1.$ donc $\|B\|_{op} \leq \frac{1}{\|A^{-1}\|} = 1 \iff |\varepsilon| < 1$

$(A-B)^{-1} = A^{-1} + A^{-1}BA^{-1} + \dots$

Exercice 13

On note $B = -\varepsilon uu^*$

| $u \in \mathbb{C}^N$ $\quad uu^* \in \mathbb{C}^{N\times N}$
| $uu^*x = \langle u, x\rangle u$ $\quad Im(uu^*) = \mathbb{C}u, \; Ker(uu^*) = \{u\}^\perp$

$\|B\|_{op} = \|u\|^2$ $\quad \|B\|_{op} \leq \frac{1}{\|A^{-1}\|_{op}} \iff |\varepsilon| \|u\|^2 \leq \frac{1}{\|A^{-1}\|_{op}}$

$\iff |\varepsilon| \leq \frac{1}{\|u\|^2 \|A^{-1}\|_{op}}$

donc dans ce cas M_ε est inversible et

$$M_\varepsilon^{-1} = A^{-1} + A^{-1}(-\varepsilon)uu^*A^{-1} + \varepsilon^2 A^{-1}uu^*A^{-1}uu^*A^{-1}$$

$$= \sum_{n=0}^{+\infty} (-1)^n \varepsilon^n (A^{-1}uu^*)^n A^{-1}$$

$$= A^{-1} + \sum_{n=1}^{+\infty} (-1)^n \varepsilon^n (A^{-1}uu^*)^n A^{-1}$$

← notation physique

$uu^* = |u\rangle\langle u|$

$uu^*x = |u\rangle \underbrace{\langle u|x\rangle}_{\text{nombre}}$

$(A^{-1}uu^*)^n A^{-1} = A^{-1}|u\rangle \overbrace{\langle u| \cdot A^{-1}|u\rangle}^{\langle u|A^{-1}|u\rangle} \langle u|A^{-1}$

$= A^{-1}|u\rangle (\langle u|A^{-1}|u\rangle)^{n-1} \langle u|A^{-1}$

$$= A^{-1} + \sum_{n=1}^{+\infty} (-\varepsilon)^n A^{-1}|u\rangle (\langle u|A^{-1}u\rangle^{n-1}) \langle u|A^{-1}$$

$$= A^{-1} - \varepsilon \left(\sum_{n=0}^{+\infty} (-\varepsilon)^n \langle u|A^{-1}u\rangle^n\right) A^{-1}|u\rangle\langle u|A^{-1} = A^{-1} - \frac{\varepsilon}{1+\varepsilon\langle u|A^{-1}u\rangle} \underbrace{A^{-1}|u\rangle\langle u|A^{-1}}_{= v v^*}$$

Supposons que $1 + \varepsilon\langle u|A^{-1}u\rangle \neq 0$

$$(A + \varepsilon|u\rangle\langle u|)\left(A^{-1} - \frac{\varepsilon}{1+\varepsilon\langle u|A^{-1}u\rangle} A^{-1}|u\rangle\langle u|A^{-1}\right)$$

$$= AA^{-1} - \frac{\varepsilon}{1+\varepsilon\langle u|A^{-1}u\rangle} |u\rangle\langle u|A^{-1}| + \varepsilon|u\rangle\langle u|A^{-1} - \frac{\varepsilon^2}{1+\varepsilon\langle u|A^{-1}u\rangle} |u\rangle\langle u|A^{-1}u\rangle\langle u|A^{-1}|$$

$$= I_n + |u\rangle\langle A^{-1}u| \left(\frac{\varepsilon(1+\varepsilon\langle u|A^{-1}u\rangle - \varepsilon - \varepsilon^2\langle u|A^{-1}u\rangle)}{1+\varepsilon\langle u|A^{-1}u\rangle} \right)$$

- Supposons $1+\varepsilon\langle u|A^{-1}u\rangle = 0$

On cherche $x \in \mathbb{C}^N\setminus\{0\}$ tel que $M_\varepsilon x = 0$ i.e. $Ax + \varepsilon\langle u|x\rangle u = 0 \Rightarrow x \in \mathrm{Vect}(A^{-1}u)$
donc on peut choisir x de la forme $x = \lambda A^{-1}u$ avec $\lambda \in \mathbb{C}\setminus\{0\}$

$$M_\varepsilon x = 0 \iff \lambda u + \varepsilon\lambda\langle u|A^{-1}u\rangle u = 0$$
$$\underset{u\neq 0}{\iff} 1+\varepsilon\langle u|A^{-1}u\rangle = 0$$

Exercice 15

$$\langle A,B\rangle_{Fr} = Tr(A^*B)$$

Soit H hermitienne, alors $H^* = H$

donc $Tr(H^*\cdot AH) = Tr(H\cdot AH)$

Attention: AH est une notation d'une matrice et pas une multiplication des matrices.

donc $\sum_{i,j}^{N} \overline{H_{ij}}\, AH_{ij} = \sum_{i,j}^{N} H_{ji}\, AH_{ij}$

$$\iff \sum_{i,j}^{N} (\overline{H_{ij}} - H_{ji}) AH_{ij} = 0$$

Sol de prof

Soit $A \in H_N$ et $B \in AH_N$

$\langle A,B\rangle_F = Tr(A^*B)$

On prend $\langle A,B\rangle = Re(Tr(A^*B))$
produit scalaire sur le $\mathbb{R}$ espace vectoriel
$M_N(\mathbb{C})$ $\underset{\text{euclidien}}{\mathbb{R}^2} \simeq \mathbb{C}$
$\simeq (z_1,z_2) \mapsto Re(\overline{z_1}, z_2)$

$$\underset{\underset{Tr(A^*B)}{\shortparallel}}{\overline{\langle A,B\rangle_F}} = \langle B,A\rangle_F = Tr(B^*A) = Tr((-B)A^*) = -Tr(A^*B)$$

Donc, on vient de montrer que $\overline{\langle A,B\rangle_F} = -\langle A,B\rangle_F$
$\iff \langle A,B\rangle_F = 0$

$$\mathcal{M}_N(\mathbb{R}) \oplus \mathcal{M}_N(\mathbb{R}) \simeq \mathcal{M}_N(\mathbb{C})$$
$$(A, B) \mapsto A + iB$$

Frobenius euclidien $\longrightarrow$ $\mathrm{Re}\, \mathrm{Tr}(Z_2^* Z_1)$

$$\mathrm{Re}\, \mathrm{Tr}\left((A_1^T - iB_1^T)(A_2^T + iB_2^T \right) = \mathrm{Tr}\,(A_1^T A_2) + \mathrm{Tr}\,(B_1^T B_2)$$