Unitary and Orthogonal matrices

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1 Introduction

Linear algebra contains a lot of particular objects with properties that make life easier. Such an example are **orthogonal matrices** or **unitary matrices**(for comlpex $\mathbb{C}$ case). Such matrices are defined in the following way:

Definition 1.1 (Unitary Matrix) – *A matrix O is called unitary if and only if*

$$O^{-1} = O^*$$

where O^* is an adjoint matrix.

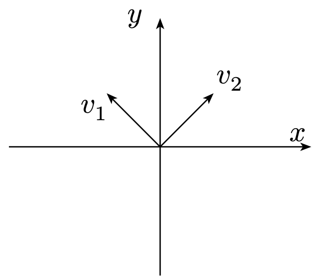
Even though, the definition may seem abstract, it has concrete geometric interpretation.

2 Intuition and Interpretation

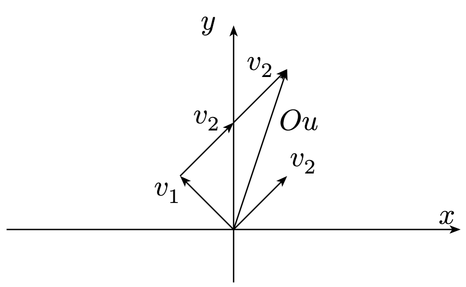
The interpretation will be given trough and example in $\mathbb{R}^2$. Let's introduce an orthogonal matrix

$$O = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = (v_1 \;\; v_2)$$

The column vectors of the matrix O are drawn on the right



Let's remind that given vector $u = \binom{x}{y}$, matrix multiplication $O \cdot u = x \cdot v_1 + y \cdot v_2$. For instance, if $u = \binom{1}{2}$, then $O \cdot u = 1 \cdot v_1 + 2 \cdot v_2$ graphically presented below:



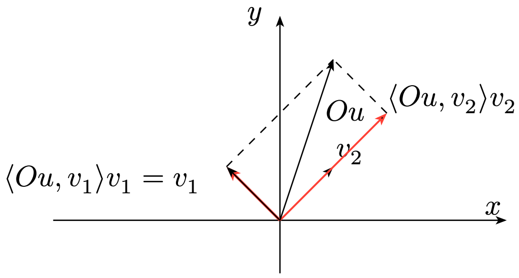
As shown on the picture, vector $Ou =: k$ is taking v_1 once, v_2 twice and gets their sum.

The important property here is the orthogonality of v_1 and v_2. To be more precise, in order to get the coordinates of a vector in an orthogonal bases it is sufficient to get dot product of the vector with each basis vector.

In our case, $O^* = O^T$, i.e. if $O = (v_1 \;\; v_2)$ then

$$O^T = \begin{pmatrix} v_1^T \\ v_2^T \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Then,



$$O^T \cdot k = \begin{pmatrix} v_1^T \cdot k \\ v_2^T \cdot k \end{pmatrix}$$

Again, as (v_1, v_2) is an orthogonal basis, dot product of k with each basis vector gives the coordinates in the basis, i.e. how much of each basis vector we have take and then sum them up. As we did previously,

$$k = O \cdot u = 1 \cdot v_1 + 2 \cdot v_2$$

Thus, $O^T k = \binom{1}{2} = u$. It is inverse operation.

Concequently, $O^{-1} = O^*$