Complex Analysis and its Applications to Linear Algebra

A self-contained introduction

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1 Introduction

This chapter develops a short but self-contained introduction to complex analysis, with a single goal in mind: arriving at the **spectral projector** formula

$$\Pi_\mu = \frac{1}{2\pi i} \oint_\gamma R_A(z)\, \mathrm{d}z, \tag{1}$$

which extracts the projector onto an eigenspace of a matrix A from a contour integral of its resolvent. Along the way we will build genuine geometric and analytic intuition — not just formulas.

The central thread is a remarkable rigidity phenomenon. A function $f : \mathbb{R} \to \mathbb{R}$ can be differentiable everywhere without having a power series representation. A function $f : \mathbb{C} \to \mathbb{C}$ that is differentiable everywhere in the complex sense **is automatically** a power series. This single fact — whose proof runs through Stokes' theorem and Cauchy's formula — is what makes complex analysis so powerful.

Roadmap. We proceed in five steps:

1. **Holomorphic functions** — what complex differentiability means geometrically (§2)
2. **Contour integrals** — integrating along curves in $\mathbb{C}$ (§3)
3. **Stokes' theorem** — why holomorphic functions have zero loop integrals (§4)
4. **Cauchy's formula** — boundary data determines interior values (§5)
5. **Application to linear algebra** — the spectral projector (§6)

2 Holomorphic Functions

2.1 The geometric meaning of complex differentiability

A function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is differentiable at p_0 if there exists a matrix $Jf(p_0)$ — the Jacobian — such that

$$f(p_0 + h) = f(p_0) + Jf(p_0) \cdot h + o(|h|). \tag{2}$$

The Jacobian can be **any** 2×2 matrix: it could represent a rotation, a shear, a reflection, an asymmetric stretch, or any combination.

The complex plane $\mathbb{C}$ carries extra structure that $\mathbb{R}^2$ does not: **multiplication**. Multiplying by a complex number $w = re^{i\theta}$ acts on $\mathbb{R}^2$ as simultaneous rotation by θ and scaling by r. As a matrix:

$$z \mapsto wz \quad \text{corresponds to} \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad w = a + ib. \tag{3}$$

This is a **very** special 2×2 matrix — it has only 2 free parameters instead of 4. The extra constraint is $a_{11} = a_{22}$ and $a_{12} = -a_{21}$.

Definition 2.1 – *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. We say f is **holomorphic** on Ω if every point $z_0 \in \Omega$ has a neighbourhood on which f is expressible as a convergent power series:*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{4}$$

Insight 2.2 – *Holomorphic = locally looks like multiplication by a complex number. Zooming in near any point z_0, the function f behaves as $f(z_0 + h) \approx f(z_0) + f'(z_0) \cdot h$ where $f'(z_0) \in \mathbb{C}$ and the multiplication is **complex** multiplication — a rotation and scaling of the input displacement h. No reflections, no shears, no asymmetric stretches.* ◇

2.2 The $\overline{\partial}$ operator and the Cauchy-Riemann equations

Writing $z = x + iy$, any smooth $f : \mathbb{C} \to \mathbb{C}$ can be thought of as depending on the two independent "coordinates" z and $\overline{z}$, since

$$x = \frac{z + \overline{z}}{2}, \quad y = \frac{z - \overline{z}}{2i}. \tag{5}$$

The chain rule gives natural differential operators in this new basis:

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \overline{\partial} \equiv \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{6}$$

These are not defined this way — they are **derived** from the change of variables. One checks immediately that $\partial_z(z^n) = nz^{n-1}$, $\overline{\partial}(z^n) = 0$, and $\overline{\partial}(\overline{z}) = 1$. So $\overline{\partial}$ is the "anti-holomorphic derivative" — it detects dependence on $\overline{z}$.

Theorem 2.3 (Cauchy-Riemann) – *Let $f = u + iv$ (with u, v real-valued) be C^1. The following are equivalent:*

1. *$\overline{\partial} f = 0$,*
2. *$\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$ (Cauchy-Riemann equations),*
3. *The Jacobian Jf has the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{R}$,*
4. *f is complex-differentiable at every point, with $f'(z_0) = a + ib$.*

Proof. The equivalence of (1) and (2) is a direct computation: $\overline{\partial} f = \frac{1}{2}(\partial_x + i\partial_y)(u + iv) = \frac{1}{2}\left[(\partial_x u - \partial_y v) + i(\partial_x v + \partial_y u)\right]$, which is zero if and only if $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$.

For (2) ⇔ (3): the Jacobian is $Jf = \begin{pmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{pmatrix}$. The Cauchy-Riemann equations are exactly the condition $\partial_x u = \partial_y v$ and $\partial_y u = -\partial_x v$, which gives $Jf = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ with $a = \partial_x u$, $b = \partial_x v$.

For (3) ⇔ (4): Jf of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ acts on $h = h_1 + ih_2$ as $(a + ib)(h_1 + ih_2)$ — precisely complex multiplication by $a + ib = f'(z_0)$. □ □

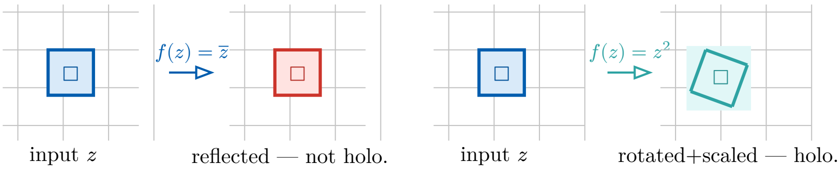

Figure 1: Left: $f(z) = \overline{z}$ reflects the plane — the Jacobian has the form $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which is **not** of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, so it is not holomorphic. Right: $f(z) = z^2$ locally rotates and scales — Jacobian has the special form, so it is holomorphic.

2.3 Examples and non-examples

The following functions are holomorphic:

- Polynomials $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ (power series terminating after finitely many terms)
- The complex exponential $z \mapsto \exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$
- The matrix exponential $z \mapsto \exp(zA)$ for fixed $A \in M_n(\mathbb{C})$
- The resolvent $\lambda \mapsto R_A(\lambda) = (A - \lambda I)^{-1}$, on $\mathbb{C} \setminus \mathrm{Sp}(A)$

The following are **not** holomorphic:

- $f(z) = \overline{z}$: gives $\overline{\partial} f = 1 \neq 0$
- $f(z) = |z|^2 = z\overline{z}$: gives $\overline{\partial} f = z \neq 0$ (except at $z = 0$)
- $f(z) = \mathrm{Re}(z) = x$: gives $\overline{\partial} f = \frac{1}{2} \neq 0$

Remark 2.4 – *$f(z) = |z|^2$ is complex-differentiable at $z = 0$ (the limit $\frac{|h|^2}{h} = \overline{h} \to 0$ as $h \to 0$), but nowhere else. Being differentiable at a single isolated point is **not** holomorphic — holomorphicity requires differentiability on an open neighbourhood.* ◇

3 Contour Integrals

3.1 Definition

Definition 3.1 (Contour integral) – *A **contour** (or **curve**) in $\mathbb{C}$ is a C^1 map $\gamma : [0, T] \to \mathbb{C}$. The **contour integral** of f along γ is defined by*

$$\int_\gamma f \, \mathrm{d}z := \int_0^T f(\gamma(t)) \cdot \gamma'(t) \, \mathrm{d}t. \tag{7}$$

This is exactly like a real integral $\int_a^b f(x) \, \mathrm{d}x$, except that:

- We sum over a **curve** in $\mathbb{C}$ rather than a segment in $\mathbb{R}$
- At each point, we multiply $f(\gamma(t))$ by $\gamma'(t)$ — the **velocity** of the curve, a complex number encoding both speed and direction
- The result is a complex number

The factor $\gamma'(t) \, \mathrm{d}t$ is the infinitesimal displacement $\mathrm{d}z$ along the curve. On a real segment $\gamma(t) = t$, we have $\gamma'(t) = 1$ and the definition reduces to an ordinary integral.

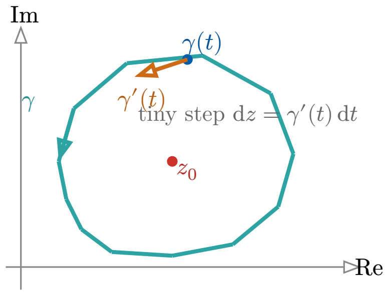

Figure 2: A contour γ in $\mathbb{C}$. At each point $\gamma(t)$, the velocity $\gamma'(t)$ gives the direction and speed of travel. The contour integral accumulates the products $f(\gamma(t)) \cdot \gamma'(t)\,\mathrm{d}t$ — tiny complex contributions — around the whole curve.

3.2 Path independence and closed loops

Two paths γ_1, γ_2 from z_0 to z_1 give the same integral when

$$\int_{\gamma_1} f\,\mathrm{d}z = \int_{\gamma_2} f\,\mathrm{d}z \iff \int_{\gamma_1} f\,\mathrm{d}z - \int_{\gamma_2} f\,\mathrm{d}z = 0. \tag{8}$$

But "γ_1 followed by γ_2 reversed" is a **closed loop**. So path independence is equivalent to: **every closed-loop integral is zero**:

$$\oint_{\gamma} f\,\mathrm{d}z = 0 \quad \text{for every closed } \gamma. \tag{9}$$

3.3 A fundamental example: the winding integral

Let $z_0 \in \mathbb{C}$ and $\gamma_\varepsilon : \theta \mapsto z_0 + \varepsilon e^{i\theta}$, $\theta \in [0, 2\pi]$ be the circle of radius ε around z_0. Then $\gamma_{\varepsilon'}(\theta) = i\varepsilon e^{i\theta}$ and:

$$\oint_{\gamma_\varepsilon} \frac{1}{z - z_0}\,\mathrm{d}z = \int_0^{2\pi} \frac{1}{\varepsilon e^{i\theta}} \cdot i\varepsilon e^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} i\,\mathrm{d}\theta = 2\pi i. \tag{10}$$

The ε cancels completely — the answer $2\pi i$ is **independent of the radius**. This is the most important integral in complex analysis.

Remark 3.2 – *The function $\frac{1}{z-z_0}$ acts like a "vortex" centered at z_0: it is holomorphic everywhere except at z_0 itself, and integrating around z_0 always gives $2\pi i$ regardless of the loop shape, as long as it winds once around z_0.* ◇

4 Stokes' Theorem and Zero Loop Integrals

4.1 The curl and Stokes' theorem

For a vector field $\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y$ on $\mathbb{R}^2$, the **scalar curl** (rotationnel) measures local rotation:

$$\operatorname{rot}\left(\vec{A}\right) = \partial_x A_y - \partial_y A_x. \tag{11}$$

Geometrically: place a tiny paddle wheel at a point in the fluid. If the flow makes it spin, the curl is nonzero there. If the flow is locally "balanced" — as much coming from one side as the other — the curl is zero and the wheel stays still.

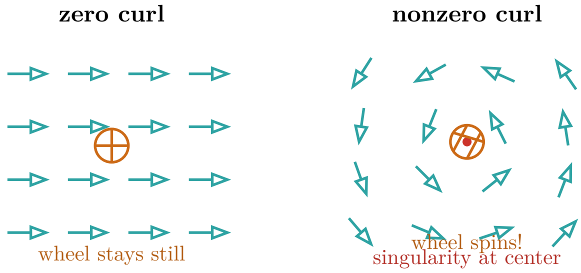

Figure 3: Left: uniform flow has zero curl — a paddle wheel placed anywhere stays still. Right: a vortex has nonzero curl — the paddle wheel spins. The curl measures local rotation of the fluid.

Theorem 4.1 (Stokes / Green) – *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\gamma = \partial\Omega$ (oriented counterclockwise), and $\vec{A} : \overline{\Omega} \to \mathbb{R}^2$ a C^1 vector field. Then*

$$\oint_{\partial\Omega} \vec{A} \cdot \mathrm{d}\vec{T} = \iint_{\Omega} \operatorname{rot}\left(\vec{A}\right) \mathrm{d}x \, \mathrm{d}y. \tag{12}$$

The proof idea: tile Ω by tiny squares of side ε. On each square, contributions from shared interior edges cancel (opposite orientations). What remains is the outer boundary. On each tiny square the integrand is approximately constant, giving a contribution $\approx \operatorname{rot}\left(\vec{A}\right) \cdot \varepsilon^2$.

4.2 Connecting to complex integrals

Writing the complex integral $\int_\gamma f \, \mathrm{d}z$ in real coordinates with $f(x, y)$ and $\vec{A}(x, y) = f(x, y)\vec{e}_x + if(x, y)\vec{e}_y$, one computes:

$$\operatorname{rot}\left(\vec{A}\right) = i\partial_x f - \partial_y f = i\left(\partial_x f + i\partial_y f\right) = 2i\overline{\partial} f. \tag{13}$$

Proposition 4.2 – *Let $f : \Omega \to \mathbb{C}$ be C^1 with $\overline{\partial} f = 0$. Then for every closed curve γ in Ω,*

$$\oint_\gamma f \, \mathrm{d}z = 0. \tag{14}$$

Proof. By Stokes, $\oint_\gamma f \, \mathrm{d}z = \iint_{\text{interior}} \operatorname{rot}\left(\vec{A}\right) \mathrm{d}x \, \mathrm{d}y = \iint 2i\overline{\partial} f \, \mathrm{d}x \, \mathrm{d}y = 0$. □

Insight 4.3 – *Why does $\overline{\partial} f = 0$ give zero loop integrals? Because holomorphic functions* ***locally look like rotation and scaling****. Walking around any loop and multiplying each step by a local rotation+scaling, the contributions from opposite sides of the loop cancel perfectly. Non-holomorphic functions break this symmetry — they can stretch differently in different directions, and the cancellation fails. The curl $\operatorname{rot}\left(\vec{A}\right) = 2i\overline{\partial} f$ measures exactly this asymmetry.* ⋄

4.3 The tiny-square computation

Let us verify the key estimate explicitly. Consider a square with corner at $z_0 = x_0 + iy_0$ and side ε. Walking counterclockwise, the four sides contribute:

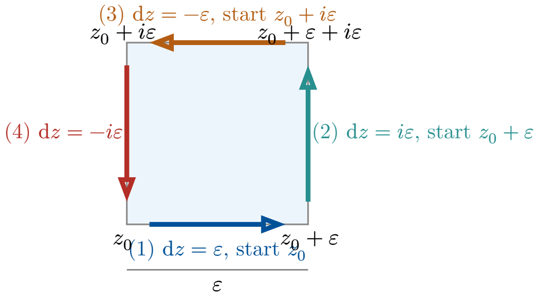

Figure 4: The four sides of a tiny square of side ε with corner at z_0, traversed counterclockwise. Each side contributes $f(\text{corner}) \times dz$ to the contour integral.

The sum of all four contributions is:

$$f(z_0) \cdot \varepsilon + f(z_0 + \varepsilon) \cdot i\varepsilon + f(z_0 + i\varepsilon) \cdot (-\varepsilon) + f(z_0) \cdot (-i\varepsilon). \tag{15}$$

Grouping and using $f(z_0 + \varepsilon) \approx f(z_0) + \varepsilon\partial_x f$ and $f(z_0 + i\varepsilon) \approx f(z_0) + i\varepsilon\partial_y f$:

$$= \varepsilon^2 \left[i\partial_x f - \partial_y f \right] = \varepsilon^2 \cdot i\left(\partial_x f + i\partial_y f\right) = 2i\varepsilon^2 \overline{\partial} f(z_0). \tag{16}$$

So $\oint_{\partial\square} f \, dz \approx 2i\varepsilon^2 \overline{\partial} f(z_0)$. If $\overline{\partial} f = 0$, every square contributes zero, and the full loop integral vanishes.

5 Cauchy's Formula

5.1 Statement

Theorem 5.1 (Cauchy's Integral Formula) – *Let $f : \Omega \to \mathbb{C}$ be C^1 with $\overline{\partial} f = 0$, and let $z_0 \in \Omega$. For any closed curve γ that winds once counterclockwise around z_0 and lies entirely in Ω:*

$$\oint_\gamma \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0). \tag{17}$$

Equivalently:

$$f(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z)}{z - z_0} \, dz. \tag{18}$$

Insight 5.2 – *This formula says something astonishing: the value of a holomorphic function at a single interior point z_0 is **completely determined** by its values on the boundary loop γ. In real analysis, a function's values at interior points are independent of boundary values. For holomorphic functions, the boundary values rigidly control everything inside — like a soap film stretched across a wire loop, where the wire (boundary) determines the film's shape (interior) completely.* ◇

Proof. The function $g(z) = \frac{f(z)}{z - z_0}$ is holomorphic on $\Omega \setminus \{z_0\}$ but has a singularity at z_0.

Step 1: deformation. Since g is holomorphic in the annular region between γ and any small circle γ_ε of radius ε around z_0, Stokes gives:

$$\oint_{\gamma} g \, \mathrm{d}z = \oint_{\gamma_\varepsilon} g \, \mathrm{d}z. \tag{19}$$

The big loop can be shrunk to a tiny loop without changing the integral.

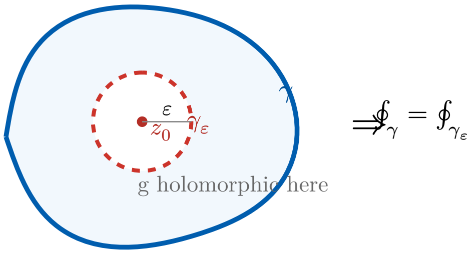

Figure 5: Deformation of contour. Since $g = f/(z - z_0)$ is holomorphic in the shaded annular region between γ and the small circle γ_ε, both integrals are equal. We can shrink γ to the tiny circle without changing the integral.

Step 2: approximation. On γ_ε, as $\varepsilon \to 0$ we have $f(z) \to f(z_0)$, so:

$$\oint_{\gamma_\varepsilon} \frac{f(z)}{z - z_0} \, \mathrm{d}z \approx f(z_0) \oint_{\gamma_\varepsilon} \frac{1}{z - z_0} \, \mathrm{d}z = f(z_0) \cdot 2\pi i, \tag{20}$$

where the last integral was computed explicitly in §3. The error in the approximation is $O(\varepsilon)$ and vanishes as $\varepsilon \to 0$, completing the proof. □

5.2 From Cauchy's formula to power series

Cauchy's formula is a **power series generator**. Fix z_0 and write w for a nearby point. Expand the Cauchy kernel in a geometric series:

$$\frac{1}{z - w} = \frac{1}{(z - z_0) - (w - z_0)} = \frac{1}{z - z_0} \sum_{n=0}^{\infty} \frac{(w - z_0)^n}{(z - z_0)^n}. \tag{21}$$

Substituting into Cauchy's formula and swapping the sum and integral (justified by uniform convergence):

$$f(w) = \sum_{n=0}^{\infty} \underbrace{\left[\frac{1}{2\pi i} \oint_{\gamma} \frac{f(z)}{(z - z_0)^{n+1}} \, \mathrm{d}z \right]}_{=: a_n} (w - z_0)^n. \tag{22}$$

This is a genuine power series for f around z_0 — proving that complex differentiability everywhere **forces** f to be a power series. This is the miracle that has no analogue in real analysis.

6 Application: the Spectral Projector

We now arrive at the linear algebra payoff. Throughout this section $A \in M_n(\mathbb{C})$.

6.1 The resolvent

Definition 6.1 – *The **spectrum** of A is* $\mathrm{Sp}(A) = \{\mu \in \mathbb{C} \mid A - \mu I \text{ not invertible}\}$. *For* $\lambda \notin \mathrm{Sp}(A)$, *the **resolvent** is*

$$R_A(\lambda) = (A - \lambda I)^{-1}. \tag{23}$$

The resolvent is a matrix-valued function of λ. It is holomorphic on $\mathbb{C} \setminus \mathrm{Sp}(A)$: by the Neumann series, for $|\lambda - \lambda_0| < 1/\left\|R_A(\lambda_0)\right\|$,

$$R_A(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_0)^n R_A(\lambda_0)^{n+1}. \tag{24}$$

The singularities of $R_A(\lambda)$ are precisely the eigenvalues of A.

6.2 The spectral projector

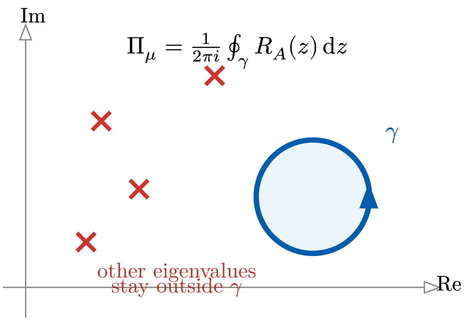

Figure 6: The contour γ winds once around the target eigenvalue μ (blue cross), while all other eigenvalues (red crosses) remain outside γ. The contour integral of the resolvent extracts the spectral projector onto the eigenspace E_μ.

Proposition 6.2 (Spectral Projector) – *Let $A \in M_n(\mathbb{C})$ be diagonalisable, μ an eigenvalue of A, and γ a closed curve winding once counterclockwise around μ but not around any other eigenvalue. Then*

$$\Pi_\mu := \frac{1}{2\pi i} \oint_\gamma R_A(z)\,\mathrm{d}z \tag{25}$$

is the projector onto the eigenspace $E_\mu = \ker(A - \mu I)$, parallel to the sum of all other eigenspaces.

Proof. Since A is diagonalisable, write $A = P^{-1}\begin{pmatrix} \mu & 0 \\ 0 & D' \end{pmatrix}P$ where $\mu \notin \mathrm{Sp}(D')$.

The resolvent block-diagonalises as:

$$R_A(z) = (A - zI)^{-1} = P^{-1}\begin{pmatrix} (\mu - z)^{-1} & 0 \\ 0 & (D' - zI)^{-1} \end{pmatrix}P. \tag{26}$$

Integrating over γ:

$$\frac{1}{2\pi i}\oint_\gamma R_A(z)\,\mathrm{d}z = P^{-1}\begin{pmatrix} \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{\mu - z} & 0 \\ 0 & \frac{1}{2\pi i}\oint_\gamma (D' - zI)^{-1}\,\mathrm{d}z \end{pmatrix}P. \tag{27}$$

Top-left block: By the winding integral of §3,

$$\frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{\mu - z} = \frac{1}{2\pi i}\oint_\gamma \frac{\mathrm{d}z}{\mu - z} = 1. \tag{28}$$

Bottom-right block: $(D' - zI)^{-1}$ is holomorphic **inside** γ (since no eigenvalue of D' lies inside γ). By Proposition 3.1, its integral over γ is zero.

Therefore:

$$\frac{1}{2\pi i}\oint_\gamma R_A(z)\,\mathrm{d}z = P^{-1}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}P = \Pi_\mu.\blacksquare \tag{29}$$

□

6.3 The functional calculus

The spectral projector is a special case of a much more general construction. For any function f holomorphic in a neighbourhood of $\mathrm{Sp}(A)$, define:

$$f(A) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - A}\,\mathrm{d}z, \tag{30}$$

where C is any contour enclosing all of $\mathrm{Sp}(A)$. This is the **holomorphic functional calculus**.

Insight 6.3 – *This extends the natural definition $f(A) = \sum a_n A^n$ (for f a power series) to* ***all*** *holomorphic f, regardless of convergence issues. Need $(I + A)^{-1}$ when $\|A\| > 1$? Write it as a contour integral. Need $\log(A)$ for a matrix with no zero eigenvalue? Use the functional calculus. The contour integral avoids all radius-of-convergence limitations.* ◇

The key properties are:
- $\mathrm{Id}(A) = A$ (the identity function gives back A)
- $(fg)(A) = f(A)g(A)$ (multiplicativity)
- If $f(z) = \sum a_n z^n$ converges and $\|A\| < \text{radius}$, then $f(A) = \sum a_n A^n$ (consistency with power series)

7 Summary: The Logical Chain

The whole chapter follows one thread:

> $\bar{\partial} f = 0 \;\Rightarrow\;$ **Stokes** $\;\Rightarrow\; \oint_\gamma f\,\mathrm{d}z = 0 \;\Rightarrow\;$ **path independence**
>
> $\Rightarrow$ **Cauchy** $\;\Rightarrow\; f(z_0) = \frac{1}{2\pi i}\oint_\gamma \frac{f(z)}{z - z_0}\,\mathrm{d}z$
>
> $\Rightarrow$ **power series** $\;\Rightarrow\;$ **spectral projector** $\Pi_\mu = \frac{1}{2\pi i}\oint_\gamma R_A(z)\,\mathrm{d}z$

Each arrow corresponds to one section of this chapter. The geometric intuition behind each step:

Step	Formula	Geometric meaning
§2	$\bar{\partial} f = 0$	Jacobian = rotation + scaling; no shear, no reflection
§3	$\int_\gamma f\,\mathrm{d}z$	Accumulate $f \times$ velocity along a curve in $\mathbb{C}$

Step	**Formula**	**Geometric meaning**
§4	$\oint f \, \mathrm{d}z = 0$	Uniform rotation+scaling cancels around any loop
§5	Cauchy's formula	Boundary values rigidly determine interior values
§5	Power series	Cauchy's kernel expands into geometric series
§6	Spectral projector	Contour around μ picks out only the μ-eigenspace

The deepest insight is that $\overline{\partial} f = 0$ — "no anti-holomorphic part" — is simultaneously four things: a condition on the Jacobian matrix, an absence of curl, path independence of integrals, and the rigid link between boundary and interior values that makes the spectral projector possible.