Notes from the Analysis and Geometry course

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Abstract

These are the notes taken during the OLMA251 - Analysis and Geometry course taught by Professor Christian Gérard. These notes contain information taken during the CMs (lectures), as well as my opinion, understanding, and things learned outside of this course.

These notes are translated into Ukrainian and English using the tool `sci-trans-git` [3]

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CHAPTER 1

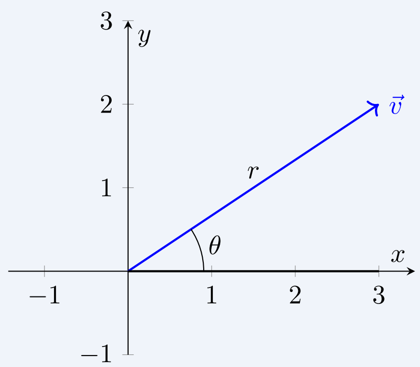
Introduction

1.1 Spaces $\mathbb{R}^d$ $\mathbb{C}^d$

Definition 1.1.
$$\mathbb{R}^d = \{X = (x_1, \ldots, x_d), x_i \in \mathbb{R}\}$$
$x_1, \ldots, x_d$ Cartesian coordinates of X

Example 1.2. $d = 2$ polar coordinates:

$$\begin{aligned}
&x = r\cos\theta \\
&y = r\sin\theta \\
&0 \le r \le \infty \quad \theta \in [0, 2\pi[
\end{aligned}$$



Definition 1.3. $\mathbb{R}^d$ is a vector space over $\mathbb{R}$

$$\begin{aligned}
&\vec{X} + \vec{Y} = (x_1 + y_1, \ldots, x_d + y_d) \\
&\lambda X = (\lambda x_1, \ldots, \lambda x_d) \quad \lambda \in \mathbb{R} \\
&\vec{0}_d = \vec{0} = (0, \ldots, 0)
\end{aligned}$$

Definition 1.4. A **scalar product**:

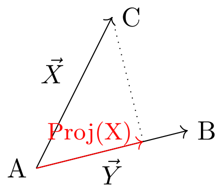
$$X \cdot Y = x_1 y_1 + x_2 y_2 + \ldots x_d y_d = \|X\|\|Y\|\cos(\theta) \text{ (where } \theta \text{ is an angle between } X \text{ and } Y\text{)}$$

Intuition. This product tells us *how closely the vectors point in the same direction* (cosine tends to 1 when θ tends to 0º, and cosine tends to 0 when θ tends to 90º). And this product allows us to have a projection of X

onto Y by the formula:

$$Proj(X) = \frac{X \cdot Y}{\|Y\|} \cdot \frac{Y}{\|Y\|}$$

$X \cdot Y$ gives the length of X and Y together, by dividing this length by $\|Y\|$ (the length of Y) we obtain the length of X on Y, we still need to multiply this length by a unit vector (of length 1) that points in the same direction as Y, (we obtain it by $\frac{Y}{\|Y\|}$)



Proposition 1.5. Scalar product respects these properties:

1. bilinearity $\quad \lambda \in \mathbb{R}$
   (a) $(X+Y) \cdot Z = X \cdot Z + Y \cdot Z$
   (b) $(\lambda X) \cdot Z = \lambda(X \cdot Z)$
   (c) $Z \cdot (X+Y) = Z \cdot X + Z \cdot Y$
   (d) $Z \cdot (\lambda X) = \lambda(Z \cdot X)$
2. symmetry $X \cdot Y = Y \cdot X$
3. positive definite: $X \cdot X \geq 0$ et $X \cdot X = 0 \Leftrightarrow X = 0_d$

Proposition 1.6. Cauchy-Schwarz:

$$|X \cdot Y| \leq (X \cdot X)^{\frac{1}{2}} (Y \cdot Y)^{\frac{1}{2}}$$

Definition 1.7. The **Euclidean norm** of a vector X is denoted:

$$\|X\| = \left(\sum_{n=1}^{d} x_i^2 \right)^{\frac{1}{2}} = \sqrt{x_1^2 + \ldots + x_d^2} = (X \cdot X)^{\frac{1}{2}}$$

often denoted $\|X\|_2$

Intuition. By the Pythagorean theorem, it is a length of this vector.

Proposition 1.8. The norm follows these properties:

1. $\|\lambda X\| = |\lambda| \|X\| \; X \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$
2. $\|X + Y\| \leq \|X\| + \|Y\|$ (triangle inequality)
3. $\|X\| \geq 0$ and $\|X\| = 0 \Leftrightarrow X = 0_d$

Proof. of (2)

$$\begin{aligned} \|X+Y\|^2 &= (X+Y) \cdot (X+Y) = X \cdot (X+Y) + Y \cdot (X+Y) = X \cdot X + X \cdot Y + Y \cdot X + Y \cdot Y \\ &= \|X\|^2 + 2X \cdot Y + \|Y\|^2 \leq \|X\|^2 + 2\|X\|\|Y\| + \|Y\|^2 = (\|X\| + \|Y\|)^2 \end{aligned}$$

Definition 1.9. A norm on $\mathbb{R}^d$ is a map $N:\ \mathbb{R}^d \to \mathbb{R}$ such that:

1. $N(\lambda X) = |\lambda| N(X)$
2. $N(X+Y) \leq N(X) + N(Y)$
3. $N(X) \geq 0$ and $N(X) = 0 \Leftrightarrow X = 0_d$

Example 1.10.

$$\|X\|_1 = \sum_{n=1}^{d} |x_i|$$
$$\|X\|_\infty = \underset{1\leq i\leq n}{max} |x_i|$$

1.2 Space $\mathbb{C}^d$

Definition 1.11.

$$\mathbb{C}^d = \{X = (x_1, \dots, x_d) :\ x_i \in \mathbb{C}\}$$

$$\begin{aligned}
&z \in \mathbb{C} \quad \overline{z} = a - ib \quad \overline{z}z = a^2 + b^2 \quad |z| = \sqrt{\overline{z}z} = \sqrt{a^2+b^2} \\
&z = a + ib \qquad a = Re\, z,\ b = Im\, z \\
&Re\, X = (Re\, x_1, \dots, Re\, x_d) \in \mathbb{R}^d \\
&Im\, X = (Im\, x_1, \dots, Im\, x_d) \in \mathbb{R}^d \\
&\underset{\in\mathbb{C}^d}{X} = \underset{\in\mathbb{R}^d}{Re\, X} + i\, \underset{\in\mathbb{R}^d}{Im\, X}
\end{aligned}$$

$\mathbb{C}^d$ is a vector space over $\mathbb{C}$ (same formulas with $\lambda \in \mathbb{C}$ field of scalars)

Definition 1.12. Scalar product:

$$(X|Y) = \sum_{n=1}^{d} \overline{x_i} y_i \in \mathbb{C}$$

Proposition 1.13. .

1. $(X|Y)$ is "linear with respect to Y"
   - $(Z|X+Y) = (Z|X) + (Z|Y)$
   - $(Z|\lambda X) = \lambda(Z|X) \quad \lambda \in \mathbb{C}$
   - $(Z|\lambda X + \mu Y) = \lambda(Z|X) + \mu(Z|Y)$
   - $(X+Y|Z) = (X|Z) + (Y|Z)$
   - $(\lambda X|Z) = \overline{\lambda}(X|Z) \quad \lambda \in \mathbb{C}$
   - $(\lambda X + \mu Y|Z) = \overline{\lambda}(X|Z) + \mu(Y|Z)$
2. $(Y|X) = \overline{(X|Y)}$
3. $(X|X) = \sum_{n=1}^{d} \overline{x_i} x_i = \sum_{n=1}^{d} |x_i|^2$
   $(X|X) \geq 0$ and $(X|X) = 0 \Leftrightarrow X = 0_d$

Proof. We have Cauchy-Schwarz:

$$(X|Y) \leq (X|X)^{\frac{1}{2}}(Y|Y)^{\frac{1}{2}}$$

same proof as before

We set:

$$\begin{aligned}&\|X\|(\text{or } \|X\|_2)\\ &= (X|X)^{\frac{1}{2}} = \left(\sum_{n=1}^{d} |x_i|^2\right)^{\frac{1}{2}}\end{aligned}$$

Hilbertian norm

$$\underset{\in \mathbb{C}^d}{\|X\|^2} = \underset{\in \mathbb{R}^d}{\|Re\, X\|^2} + i \underset{\in \mathbb{R}^d}{\|Im\, X\|^2}$$

Lemma 1.14.

$$\|X\| = \underset{\|Y\| \leq 1}{sup} |(X|Y)|$$

Proof. $|(X|Y)| \leq \|X\|\|Y\| \leq \|X\|$ si $\|Y\| \leq 1$

$$\underset{\|Y\| \leq 1}{sup} |(X|Y)|$$

Other meaning:

$$\begin{aligned}&X \neq 0 \quad Y = \frac{X}{\|X\|} = \lambda X \quad \lambda = \frac{1}{\|X\|}\\ &\|Y\| = |\lambda|\|X\| = \frac{1}{\|X\|}\|X\| = 1\\ &(X|Y) = (X|\frac{X}{\|X\|}) = \frac{1}{\|X\|}(X|X) = \|X\|\\ &sup\{|(X|Y)| : \|Y\| \leq 1\}\\ &\|X\| \leq sup\{|(X|Y)| : \|Y\| \leq 1\} \quad (\text{take } Y = \frac{X}{\|X\|})\end{aligned}$$

Other norms on $\mathbb{C}^d$

- $\|X\|_1 = \sum_{n=1}^{d} |x_i| \quad X \in \mathbb{C}^d$
- $\|X\|_\infty = \underset{1 \leq i \leq d}{sup} |x_i|$

1.3 Distance on $\mathbb{R}^d$

We forget norm and scalar product. We introduce the distance

Definition 1.15. A distance is a mapping:

$$\begin{aligned}d : \mathbb{R}^d &\longrightarrow \mathbb{R}\\ (X, Y) &\longmapsto d((X, Y))\end{aligned}$$

that follows these properties:

1. $d(X, Y) = d(Y, X)$ (symmetry)

2. $d(X,Y) \leq d(X,Z) + d(Z,Y)$ (triangle inequality) $\forall X,Y,Z$
3. $d(X,Y) \geq 0 \quad \forall X,Y$ and $d(X,Y) = 0 \Leftrightarrow X = Y$

Definition 1.16. The Euclidean distance

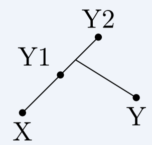
$$d(X,Y) = \|X - Y\| = \sqrt{\sum_{n=1}^{d} (x_i - y_i)^2}$$

Example 1.17. Distances

1. $d_2(X,Y) = \|X - Y\|_2$ (Euclidean distance on $\mathbb{R}^d$)
2. $d_1(X,Y) = \|X - Y\|_1$
   $d_\infty(X,Y) = \|X - Y\|_\infty$
3. logarithmic distance on $\mathbb{R}_+$: $d(a,b) = |b - a|$

$$\log_{10}(a) = \frac{\log(a)}{\log(10)}$$

   $x, y \in]0, +\infty[$
   $d_{\log}(x,y) = |\log_{10}(\frac{y}{x})|$
   i is a distance on $]0, +\infty[$
   $d_{\log}(100, 110) = \log_{10}(1,1)$
4. SNCF distance



   $d(X,Y)$ usual distance in $\mathbb{R}^2$ we define:

$$\delta(X,Y) = \begin{cases} d(X,Y) \text{ if } X,0,Y \text{ aligned} \\ d(X,0) + d(0,Y) \text{ otherwise} \end{cases}$$

Proposition 1.18. Let E be a metric space and two distances d_1 and d_2. The distances are said to be **equivalent** if $\exists a, b \in \mathbb{R}$ such that:

$$\forall x, y \in E, \quad a \cdot d_1(x,y) \leq d_2(x,y) \leq b \cdot d_1(x,y)$$

CHAPTER 2

Metric Spaces

Definition 2.1. E equipped with a distance application d (see Definition 1.15) is denoted (E, d): metric space

Remark 2.2. if $d_1 \neq d_2$ (E, d_1) has nothing to do with (E, d_2)

Remark 2.3. Remember the following version of the triangle inequality:

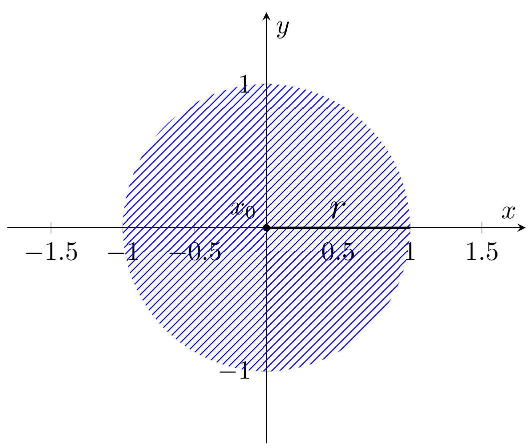
$$|d(x, z) - d(y, z)| \leq d(x, y)$$

Remark 2.4. Induced distance:
If (E, d) is a metric space and $U \subset E$. I can restreidnre d to $U \times U$: (U, d) is also a metric space.

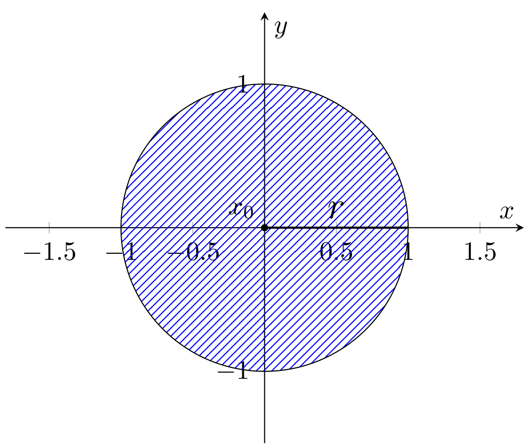
2.1 Balls in a metric space

Definition 2.5. (E, d) metric space. Let $x_0 \in E$ and $r \geq 0$

1. $B(x_0, r) = \{x \in E : d(x_0, x) < r\ \}$ open ball with center x_0, radius r
2. $B_f(x_0, r) = \{x \in E : d(x_0, x) \leq r\}$ closed ball with center x_0, radius r



(a) open balls (i.e $d(x_0, x) < r$)



(b) closed balls (i.e $d(x_0, x) \leq r$)

Lemma 2.6. .

1. $B(x_0, 0) = \emptyset$ (because it's impossible to have points whose distance is strictly less than 0)
2. $B_f(x_0, 0) = \{x_0\}$
3. $B(x_0, r_1) \subset B_f(x_0, r_1) \subset B(x_0, r_2)$ if $r_1 < r_2$
4. $B(x_1, r_1) \subset B(x_0, r)$ if $d(x_0, x_1) + r_1 \leq r$

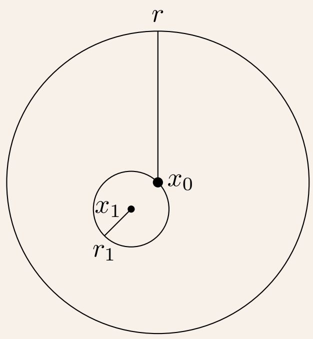


Figure 2.2: Lemma 4

Proof. I assume that $d(x_0, x_1) \leq r$
Let $x \in B(x_1, r_1)$ therefore $d(x_1, x) < r_1$ to show: $x \in B(x_0, r)$ (i.e $d(x_0, x) < r$?)
The triangle inequality states:

$$\begin{aligned} d(x_0, x) &\leq d(x_0, x_1) + d(x_1, x) \\ &< d(x_0, x_1) + r_1 \leq r \\ &\Rightarrow x \in B(x_0, r) \end{aligned}$$

Example 2.7. 1. $E = \mathbb{R}, \quad d(x, y) = |x - y|$

$$B(x_0, r) =]x_0 - r, x_0 + r[$$

2. $E = \mathbb{R}^d, \quad d = 2, 3, \quad X = (x_1, \dots, x_d)$

$$\begin{aligned} \|X\|_2 &= \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}} \\ \|X\|_1 &= \sum_{i=1}^{d} x_i \\ \|X\|_\infty &= \max_{1 \leq i \leq d} |x_i| \end{aligned}$$

$$\begin{aligned} &d_2(X, Y) = \|Y - X\|_2 = \|\vec{XY}\|_2 \\ &d_1(X, Y), d_\infty(X, Y) \end{aligned}$$

Property. In $\mathbb{R}^n$

- $d_\infty(X, Y) \leq d_1(X, Y) \leq n d_\infty(X, Y)$
- $d_\infty(X, Y) \leq d_2(X, Y) \leq \sqrt{n} d_\infty(X, Y)$

2.2 Bounded sets of (E, d)

Definition 2.8. Let $A \subset E$. A is bounded if $\exists R > 0$ and $\exists x_0 \in E$ such that

$$A \subset B(x_0, R)$$

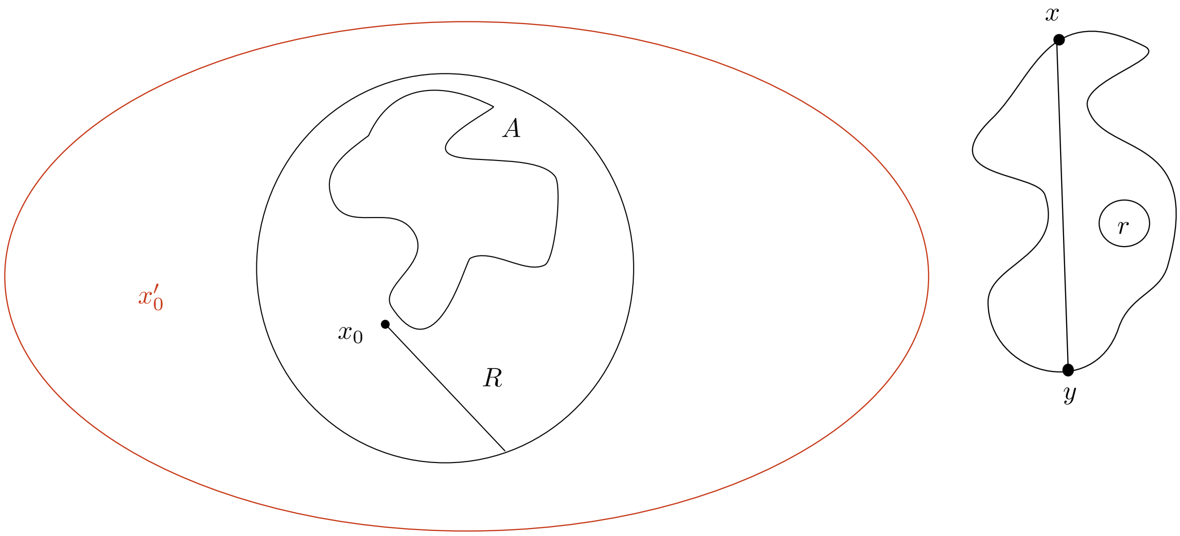


Figure 2.3: Example of a bounded set

Lemma 2.9. The following properties are equivalent:

1. A is bounded
2. $\forall x_0 \in E, \exists r > 0$ such that $A \subset B(x_0, r)$
3. $\exists r > 0$ such that $\forall x, y \in A$ we have $d(x, y) < r$

Proof. of lemma

- $(1) \Rightarrow (2)$:
  Hypothesis: $\exists x_1 \in E, \exists r_1 \in E$ such that $A \subset B(x_1, r_1)$
  Let $x_0 \in E$. Goal: to find r such that $A \subset B(x_0, r)$ if $x \in A$, we have: $d(x_1, x) < r_1$
  I want: $d(x_0, x) < r$

$$d(x_0, x) \leq d(x_0, x_1) + d(x_1, x) \leq d(x_0, x_1) + r_1 < r \quad \text{if } r > d(x_0, x_1) + r_1$$

Property.

1. Every finite part is bounded
2. If A bounded and $B \subset A$ then B is bounded
3. The union of a finite number of bounded sets is bounded

Proof. of (3).
$A_1, \dots, A_n$ are bounded. I fix $x_0 \in E$, A_i bounded $(1 \le i \le n)$, therefore $\exists r_i > 0$ such that $A_i \subset B(x_0, r_i)$ if $r = \underset{1 \le i \le n}{max}\, r_i$

$$A_i \subset B(x_0, r), \forall i \Rightarrow \bigcup_{i=1}^{n} A_i \subset B(x_0, r)$$

□

2.3 Bounded functions

Definition 2.10. Let B be a set. A function $F : B \to E$ is bounded if $F(B) = \{F(b) : b \in B\} \subset E$ is bounded.

2.4 Distance between sets

Definition 2.11. The distance between two sets A, B is:

$$d(A, B) := \underset{x \in A, y \in B}{inf} d(x, y)$$

Intuitively, we are looking for two points x and y such that the distance is as small as possible.

Definition 2.12. The distance between a point x and a set B is:

$$d(x, B) := \underset{y \in B}{inf} d(x, y)$$

The same intuition.

Property. $\forall x \in A,\ y \in B,\ d(x, y) \ge d(A, B)$ and $\forall \varepsilon > 0, \exists x \in A,\ y \in B$ such that $d(x, y) \le d(A, B) + \varepsilon$

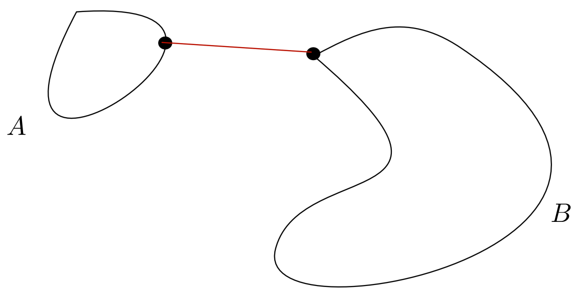


Figure 2.4: Distance between sets

2.5 Topology of metric spaces

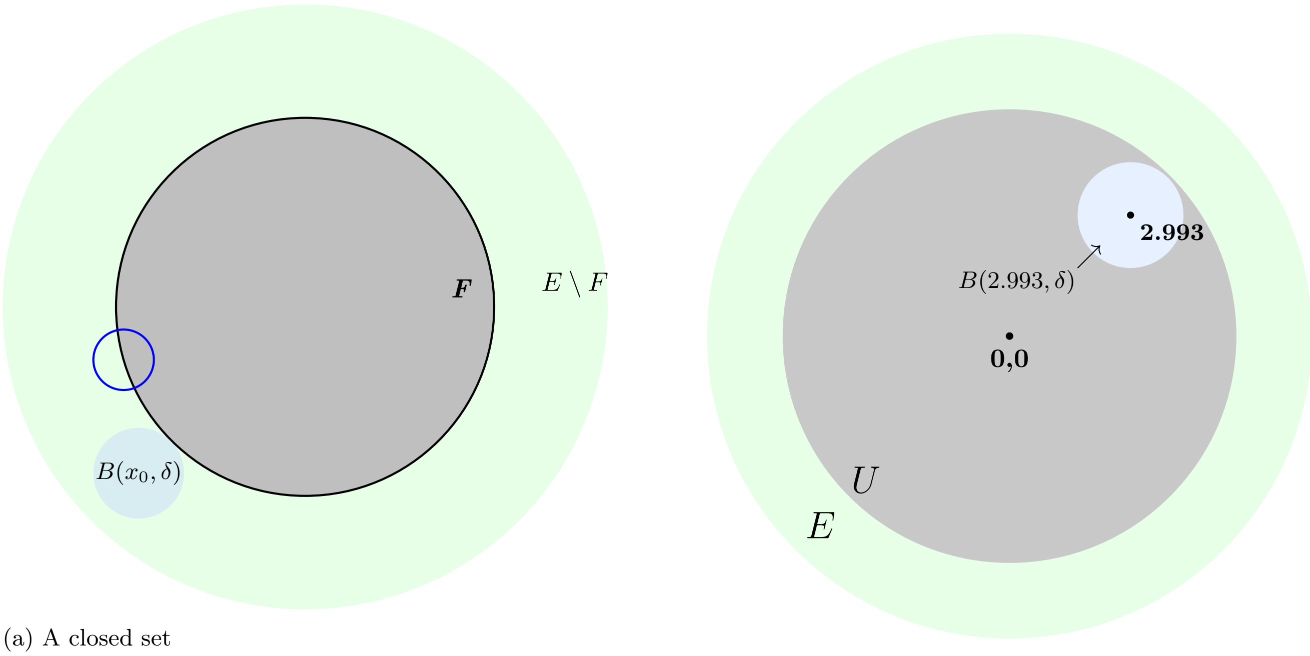
distance $d(x, y) \longrightarrow$ balls $B(x_0, r) \longrightarrow$ open sets

Definition 2.13. Let (E, d) be a metric space.

1. $U \subset E$ is open if $\forall x_0 \in U$, $\exists r > 0\, r(x_0)$ such that $B(x_0, r) \subset U$

2. $F \subset E$ is closed if $E \setminus F$ is open

$\emptyset$ is open and E is open. $\emptyset$ is closed and E is closed.



(a) A closed set
At the boundary, it is impossible to find balls that belong to F, because it is impossible to have an open ball of $r = 0$. Example: dark blue circle
For every point in $E \setminus F$ we can find an open ball

(b) An open set
for every point near the boundary we can find a ball infinitesimally small with points around this point included in U.

Figure 2.5: Demonstration of open and closed spaces

Remark 2.14. in $\mathbb{R}$ open intervals are open sets (same for closed ones)

Remark 2.15. A distance between two open sets always exists and it is the infimum (which is never reached)

Lemma 2.16. 1. $B(x_0, r_0)$ is open.

2. $B_f(x_0, r_0)$ is closed.

Proof. 1. Let $x_1 \in B(x_0, r_0)$ $(d(x_0, x_1) < r_0)$.
Goal: find $r_1 > 0$ such that $B(x_1, r_1) \subset B(x_0, r_0)$?

$$x \in B(x_1, r_1) : \; d(x_1, x) < r_1$$
$$x \in B(x_0, r_0) \text{ if } d(x_0, x) < r_0$$

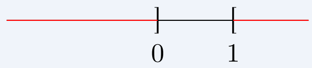
easy:

$$\begin{aligned} d(x_0, x) &\leq d(x_0, x_1) + d(x_1, x) \\ &\leq d(x_0, x_1) + r_1 \\ &< r_0 \text{ if} \end{aligned}$$

$r_1 < r_0 - d(x_0, x_1) > 0$

$\square$

Example 2.17. bizarre.
Let $E = \mathbb{R}$, $d(x, y) = |y - x|$, $A =]0, 1[$ open, not closed in $\mathbb{R}$.



I consider A as a part of (A, d). As $A \setminus A = \emptyset$ which is open, therefore A is closed in A. However, the bounds are never reached, so A is open in (A, d).

Theorem 2.18. .

1. Let U_i, $i \in I$ be a collection of open sets. Then, $\cup_{i \in I} U_i$ is open.
   Translate: Any union of open sets is open.
2. If $U_1, \dots, U_n$ are open
   $$\bigcap_{i=1}^{n} U_i \text{ is open.}$$
   Translate: <u>finite</u> intersection of open sets is open.

1. Let U_i, $i \in I$ be a collection of closed sets. Then, $\cup_{i \in I} U_i$ is closed.
   Translate: Any union of closed sets is closed.
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   $$\bigcap_{i=1}^{n} U_i \text{ is closed.}$$
   Translate: <u>finite</u> intersection of closed sets is closed.

Proof. .

1. Let $x \in U := \bigcup\limits_{i \in I} U_i$. There exists an i denoted i_0 such that $x \in U_{i_0}$, U_{i_0} is open, so $\exists r > 0$ such that $B(x, r) \subset U_{i_0} \subset U := \bigcup\limits_{i \in I} U_i$.
2. Let $x \in U := \bigcap\limits_{1 \le i \le n} U_i$.
   We fix i. $x \in U_i$, U_i open, so $\exists r_i > 0$ such that $B(x, r) \subset U_i$, $1 \le i \le n$, so $B(x, r) \subset U := \bigcap\limits_{1 \le i \le n} U_i$

□

2.6 Algorithms to show that a set is open/closed

Show that a set is open	Show that a set is closed

• **Use the definition :** $\forall x \in \mathcal{U}, \exists r > 0 \quad \text{such that} \quad B(x,r) \subset \mathcal{U}$
• Show that $E \setminus \mathcal{U}$ is closed.
• Show that $\mathcal{U}$ is the preimage of an open set by a continuous application.
• Express $\mathcal{U}$ as an open ball.
• Write $\mathcal{U}$ as :
  – a union of open sets ;
  – a finite intersection of open sets.
• $\mathcal{U} = \text{Int}(U)$.
• Write $\mathcal{U} = I_1 \times \cdots \times I_n$ with I_i open.
	• **Use the definition :** $E \setminus V$ is open.
	• **Sequential characterization :** Any convergent sequence in V, its limit is also in V.
	• Show that V is the preimage of a closed set by a continuous application.
	• Show that V is compact.

2.7 Interior, closure, boundary

2.7.1 Interior

Definition 2.19. Let $A \subset E$.

1. $x_0 \in E$ is interior to A if $\exists \delta > 0$ such that:
$$B(x_0, \delta) \subset A$$
2. $Int(A)$ (interior of A) = all points interior to A. (also denoted $\mathring{A}$)

Intuition. $Int(A)$ is a set that is entirely within A and is far from the edges of A.

Proposition 2.20. $Int(A)$ is the largest open set included in A. Equivalently, $Int(A)$ is the union of all open sets included in A.

Proof.

1. $Int(A) \subset A$: clear
2. $\underline{Int(A) \text{ is open}}$:
   Let $x_0 \in Int(A)$.

   Goal: find δ_0 such that $B(x_0, \delta_0) \subset Int(A)$. Find δ_0 such that if $d(x_0, x) < \delta_0$ then $x \in Int(A)$?

   Hyp: $x_0 \in Int(A)$. $\exists \delta_1 > 0$ such that $B(x_0, \delta_1) \subset A$. We have seen that $B(x_0, \delta_1)$ is open. I say that $B(x_0, \delta_1) \subset Int(A)$.

   Proof: Let $x \in B(x_0, \delta_1)$. $B(x_0, \delta_1)$ is open, so $\exists \delta_2 > 0$ such that $B(x, \delta_2) \subset B(x_0, \delta_1) \subset A$. So $x \in Int(A)$, so $B(x_0, \delta_1) \subset Int(A)$.

   $Int(A)$ is open.
3. If U is open and $U \subset A$ then $U \subset Int(A)$?
   $x_0 \in U$. U open $\Rightarrow \exists \delta$ such that $B(x_0, \delta) \subset U \subset A \Rightarrow x_0 \in Int(A)$

□

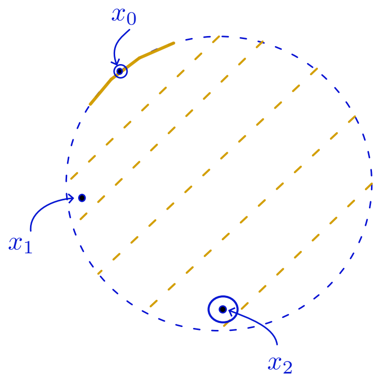

Figure 2.6: Example of an interior

2.7.2 Adherent

Definition 2.21. Let $A \subset E$.

1. $x_0 \in E$ is adherent to A, if $\forall \delta > 0$, $B(x_0, \delta)$ intersects A. (equivalent to $d(x_0, A) = 0$)
2. $Adh(A)$ (adherence or closure of A) = set of adherent points to A (also denoted $\overline{A}$)

Intuition. Closure helps complete sets. If A is open, then its boundaries do not belong to A, but they belong to $Adh(A)$.

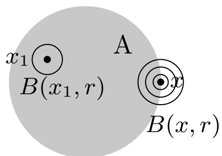

Figure 2.7: Adherent

Proposition 2.22. $Adh(A)$ is the smallest closed set that contains A (the intersection of all closed sets that contain A)

Proof.
1. $A \subset Adh(A)$ clear
2. $Adh(A)$ is closed?
   We show that $E \setminus Adh(A)$ is open.
   $x_0 \in Adh(A) \Leftrightarrow \forall \delta > 0,\ B(x_0, \delta) \cap A \neq \emptyset$
   $x_0 \notin Adh(A) \Leftrightarrow \exists \delta_0 > 0$ s.t. $B(x_0, \delta_0) \cap A = \emptyset \Leftrightarrow \exists \delta_0 > 0$ s.t. $B(x_0, \delta_0) \subset E \setminus A \Leftrightarrow x_0 \in Int(E \setminus A)$

Then:

$$E \setminus Adh(A) = Int(E \setminus A)$$
$$Adh(A) = (Int(\underbrace{A^c}_{=E\setminus A}))^c$$

□

Definition 2.23. Let $A \subset B$. We say that A is **dense** in B if $B \subset Adh(A)$
Let $x_0 \in B$, $\forall \varepsilon > 0\, \exists x_\varepsilon \in A$ such that $d(x_0, x_\varepsilon) < \varepsilon$

Example 2.24.

$$\mathbb{Q}^2 = \{(x,y) : x, y \in \mathbb{Q}\} \text{ dense in } \mathbb{R}^2$$

Definition 2.25. alternative of density. Let $A \subset B$. A is dense in B if every open ball in B contains at least one élémens of A.

2.7.3 Boundary

Definition 2.26. Let $A \subset E$. The **boundary** of A (or the boundary of A) denoted $Fr(A)$ or ∂A is:

$$Adh(A) \cap Adh(E \setminus A)$$

Example 2.27. in $\mathbb{R}$

1. $Int(\mathbb{Q}) = \emptyset$
2. $Int(\mathbb{R} \setminus \mathbb{Q}) = \emptyset$
3. $Adh(\mathbb{Q}) = \mathbb{R}$
4. $Adh(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$
5. $Fr(\mathbb{Q}) = \mathbb{R}$
6. $Fr(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$

Example 2.28. $E = \{a, b, c\}$ We set:

- $d(a,a) = d(b,b) = d(c,c) = 0$
- $d(a,b) = d(b,a) = d(b,c) = d(b,c) = 1$
- $d(a,c) = d(c,a) = 2$

$$B(a,2) = \{a,b\} = Adh(B(a,2))$$
$$B_f(a,2) = \{a,b,c\}$$

Proposition 2.29. 1. $Int(A) \subset A \subset Adh(A)$

2. $E = Int(E \setminus A) \cup Fr(A) \cup Int(A)$ (disjoint union)

3. $E \setminus Int(A) = Adh(E \setminus A)$

4. $E \setminus Adh(A) = Int(E \setminus A)$

5. $Fr(A) = Adh(A) \setminus Int(A)$

Proposition 2.30. 1. A open $\Leftrightarrow A = Int(A)$

2. A closed $\Leftrightarrow A = Adh(A)$

3. $x \in Adh(A) \Leftrightarrow d(x, A) = 0$

4. $x \in Int(A) \Leftrightarrow d(x, E \setminus A) > 0$

2.8 Sequence in a metric space

Definition 2.31. E a set. A sequence in E: denoted $(u_n)_{n\in\mathbb{N}}$ it is a function $u : \mathbb{N} \to E$ where $u(n)$ is denoted u_n is the the n^{th} term of the sequence $(u_n)_{n\in\mathbb{N}}$.

If $E = \mathbb{R}^d$

$$\mathbb{R}^d \ni X_n = (x_{1,n}, \ldots, x_{d,n})$$

where $(x_{i,n})_{n\in\mathbb{N}}$ sequences in $\mathbb{R}$

Definition 2.32. Let (x_n) be a sequence in E and $x \in E$. We say that $\lim_{n\to\infty} x_n = x$ if $\lim_{n\to\infty} d(x_n, x) = 0$.

($\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq N$, $d(x_n, x) < \varepsilon$)

Proposition 2.33. $(x_n)_{n\in\mathbb{N}}$ is bounded if $\{x_n : n \in \mathbb{N}\}(\subset E)$ is a bounded set.

Remark 2.34. in $\mathbb{R}^d$ equipped with d_2 (Euclidean distance)

$$X_n = (x_{1,n}, \ldots, x_{d,n})$$
$$X = (x_1, \ldots, x_d)$$

$$\lim_{n\to\infty} X_n = X \Leftrightarrow \lim_{n\to\infty} x_{i,n} = x_i \quad (1 \leq i \leq d)$$

Proposition 2.35. the limit of a convergent sequence is unique.

Proof.

$$\text{If } X_n \xrightarrow[n\to\infty]{} X \text{ and } X_n \xrightarrow[n\to\infty]{} X'$$

$$d(X, X') \leq \underbrace{d(X, X_n)}_{\to 0} + \underbrace{d(X_n, X')}_{\to 0} \Rightarrow d(X, X') = 0 \Rightarrow X = X'$$

□

Proposition 2.36. (link with adherence)

1. $x \in Adh(A)$ if and only if there exists a sequence (x_n) of elements from A such that $\lim_{n\to\infty} x_n = x$

2. A is closed iff for every sequence (x_n) of elements from A that converges to $x \in E$ we have $x \in A$

Intuition. 1. If $(x_n)_{n\in\mathbb{N}}$ consists of elements of A ($\forall n \in N$, $x_n \in A$), then it converges to an element x which can be either in A, or at the boundary of the elements of A, so at the frontier.

2. If the limit of any sequence $(x_n)_{n\in\mathbb{N}}$ of elements of A is also in A, then the boundary of A is included in A. Because one of the sequences tends towards the boundary.

Proof. of Prop. 2.36

1. ($\Leftarrow$) Let (x_n) with $x_n \in A \quad \forall n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = x$.
   I have $d(x_n, x) \xrightarrow[n\to\infty]{} 0$ and $x_n \in A$, so

$$inf_{y\in A}(d(x,y)) = 0 = d(x, A)$$
$$d(x, A) = 0 \Leftrightarrow x \in Adh(A)$$

   ($\Rightarrow$) Let $x \in Adh(A)$

$$\Leftrightarrow d(x, A) = 0$$
$$\Leftrightarrow \forall \varepsilon > 0, \exists x_\varepsilon \in A \text{ such that } d(x, x_\varepsilon) < \varepsilon$$

   Take $\varepsilon = \frac{1}{n}$, I set $u_n = x_{\frac{1}{n}}$. $u_n \in A$ $d(x, u_n) < \frac{1}{n}$, so $\lim_{n\to\infty} u_n = x$

2. ($\Rightarrow$) Let A be closed, so

$$A = Adh(A)$$

   If (x_n) is a sequence in A that converges to x.

$$x \in Adh(A) = A$$

   ($\Leftarrow$) We say that $Adh(A) \subset A$. Since $A \subset Adh(A)$, so $A = Adh(A)$

□

2.9 Cauchy Sequences

Definition 2.37. $(x_n)_{n\in\mathbb{N}}$ sequence in E is Cauchy if:

$$\forall \varepsilon > 0\, \exists N(\varepsilon) \in \mathbb{N} \text{ such that } \forall n, p \geq N(\varepsilon), d(x_n, x_p) \leq \varepsilon$$

Intuition. A Cauchy sequence is like measuring a point and localizing it, i.e:

1. We say it is between 0 and 1.
2. Then, we specify more and say it is between 0.5 and 0.6.
3. Then, between 0.55 and 0.56

We can infinitely increase the level of precision. That's the idea of a Cauchy sequence.

Proposition 2.38. 1. Every Cauchy sequence is bounded.

2. Every convergent sequence is Cauchy

Proof. 1. Let $(x_n)_{n\in\mathbb{N}}$ be a Cauchy sequence. Then, by definition

$$\forall \varepsilon > 0, \exists N \in \mathbb{N} \text{ such that } \forall n, p \geq N, d(x_n, x_p) < \varepsilon$$

Let $\varepsilon = 1$. So $\exists N \in \mathbb{N}$ such that $\forall n, p \geq N, d(x_n, x_p) < 1$, so $\forall n \geq N$, $d(x_n, x_N) < 1$. We therefore have:

$$\forall n \in N, d(x_n, x_N) < 1 + \overbrace{\sup_{1\leq i\leq N} d(x_n, x_N)}^{=:r_0}$$

Then $\forall n \in \mathbb{N}$, $x_n \in B(x_N, 1+r_0)$ so $(x_n)_{n\in\mathbb{N}}$ is bounded.

2. Let (x_n) be a sequence with $\lim_{n\to\infty} x_n = x$ with $x \in E$.
   - Hypothesis: $\frac{\varepsilon}{2} > 0\, \exists N(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that $\forall n \geq N(\frac{\varepsilon}{2}), d(x_n, x) \leq \varepsilon/2$
   - To show: $\varepsilon > 0\, \exists M(\varepsilon) \in \mathbb{N}$ such that $\forall n, p \geq M(\varepsilon), d(x_n, x_p) \leq \varepsilon$

$$d(x_n, x_p) < d(x_n, x) + d(x, x_p) \text{ if } n, p \geq N(\frac{\varepsilon}{2})\, d(x_n, x_p) \leq 2\frac{\varepsilon}{2} = \varepsilon$$

□

Definition 2.39. (E, d) is complete if every Cauchy sequence in E is convergent.

Definition 2.40. A metric space (E, d) is **complete** if every sequence $(x_n)_{n\in\mathbb{N}}$ of elements of E converges to a limit x which also belongs to E.

Intuition. It's not quite correct to say, but, we can say that a Cauchy sequence $(x_n)_{n\in\mathbb{N}}$ always converges because there is a moment $N \in \mathbb{N}$ after which the elements are very close but the limit does not always belong to the set in which this sequence is Cauchy.

For example, a sequence $(u_n)_{n\in\mathbb{N}}$ with values in $\mathbb{Q}$ that converges to $\sqrt{2}$ in $\mathbb{R}$. In $\mathbb{R}$ it is convergent and Cauchy, but in $\mathbb{Q}$ it is Cauchy but not convergent because the limit $\sqrt{2} \notin \mathbb{Q}$.

Example 2.41. A metric space $(]0, 1], d)$ with d a Euclidean distance is not complete, because consider a sequence: $x_n = \frac{1}{n}$ whose limit is 0. However, $0 \notin]0, 1]$. Therefore, this space is not complete.

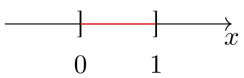

Figure 2.8: $(]0, 1], d)$ is not complete

Example 2.42. A space $(\mathbb{Q}, d)$ is not complete. Because we can take a sequence x_n tending towards $\sqrt{2} \notin \mathbb{Q}$.

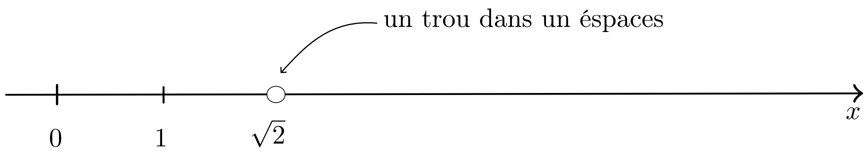

Figure 2.9: $\mathbb{Q}$ not complete

Proposition 2.43. $\mathbb{R}^d$ equipped with the usual distance is complete.

Proof.

$$X_n = (x_{1,n}, \dots, x_{d,n})$$
$$|x_i - y_i| \leq d(X, Y) = \|X - Y\|_2 \quad \forall 1 \leq i \leq d$$

the real sequences $(x_{i,n})_{n\in\mathbb{N}}$ are Cauchy if (X_n) is Cauchy. □

Property. $\mathbb{R}$ is complete

Proof. (Follows from the property of the supremum)
There exists $x_i \in \mathbb{R}$ with $1 \leq i \leq d$ such that $|x_{i,n} - x_i| \xrightarrow[n\to\infty]{} 0$

$$d(X,Y) \leq \sqrt{d} \max_{1\leq i\leq d} |x_i - y_i|$$

therefore $X_n \xrightarrow[n\to\infty]{} X$, $X = (x_1, \ldots, x_d)$ □

2.10 Subsequences

Definition 2.44. Let $(x_n)_{n\in\mathbb{N}}$ a sequence in E. A sequence

$$(y_n)_{n\in\mathbb{N}} \text{ with } y_n = x_{\phi(n)}$$

where $\phi : \mathbb{N} \to \mathbb{N}$ is strictly increasing is called a **subsequence** of the sequence (x_n).

Example 2.45. Let an application $\phi : \mathbb{N} \to \mathbb{N}$ such that $\phi(n) = 2n$. Therefore $(x_n)_{\phi(n)}$ is a subsequence of $(x_n)_{n\in\mathbb{N}}$ and:

$$(x_n)_{\phi(n)} = \{x_0, x_2, x_4, \ldots\}$$

Proposition 2.46.

1. Every subsequence of a convergent sequence converges to the limit of that sequence.
   This means that, $\forall (x_n)_{n\in\mathbb{N}}$ such that $\exists x \in E, \lim_{n\to\infty} x_n = x$
   $$\forall \phi : \mathbb{N} \to \mathbb{N} \text{ strictly increasing, } \lim_{n\to\infty} x_{\phi(n)} = x$$
2. If (x_n) is Cauchy and has a subsequence that converges to X, then (x_n) converges to x.

Proof.

1. Let (x_n) with $\lim x_n = x$
   $$\forall \varepsilon > 0 \, \exists M(\varepsilon) \text{ such that if } n \geq N(\varepsilon), d(x_n, x) \leq \varepsilon$$
   Let $y_n = x_{\phi(n)}$ a subsequence.
   - Goal: Let $\varepsilon > 0$, find $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$, $d(\underbrace{y_n}_{:=x_{\phi(n)}}, x) \leq \varepsilon$

   I choose $N(\varepsilon)$ such that if $n \geq N(\varepsilon)$ then $\phi(n) \geq M(\varepsilon)$, so $d(y_n, x) d(x_{\phi(n)}, x) \leq \varepsilon$. This is possible because $\phi(n) \xrightarrow[n\to\infty]{} \infty$, $N(\varepsilon) = M(\varepsilon)$
2. - Hyp1: $\forall \varepsilon > 0 \, \exists M(\varepsilon)$ such that if $n, p \geq M(\varepsilon)$ $d(x_n, x_p) \leq \varepsilon$
   - Hyp2: $\forall \varepsilon > 0 \, \exists P(\varepsilon)$ such that if $p \geq P(\varepsilon), d(y_p, x) \leq \varepsilon$, $d(y_p, x) = d(x_{\phi(p)}, x)$

$$d(x_n, x) \leq d(x_n, x_{\phi(p)}) + d(x_{\phi(p)}, x) \quad \text{by the triangle inequality}$$

$$d(x_n, x_{\phi(p)}) \leq \varepsilon \text{ if } n \geq M(\varepsilon) \text{ and } \phi(p) \geq M(\varepsilon)$$

$$d(x_{\phi(p)}, x) \leq \varepsilon \text{ if } p \geq P(\varepsilon)$$

If $n \geq M(\varepsilon)$, I choose p such that $\phi(p) \geq M(\varepsilon)$ and $p \geq P(\varepsilon)$. I fix this p!

$$\text{if } n \geq M(\varepsilon) \text{ then } d(x_n, x) \leq 2\varepsilon$$

□

2.11 Construction process of the interior and closure

I have $A \subset \mathbb{R}$ or $\mathbb{R}^2$ (or $\mathbb{R}^3$). I must find $Int(A)$ and $Adh(A)$

1. I draw A on a sheet
2. I think that $Int(A) = C$ (C is said to be included in A!)
   (a) I show that <u>C is open</u> (easy), therefore
   $$C \subset Int(A)$$
   because $Int(A)$ is the largest open set included in A.
   (b) I show that $Int(A) \subset C$, i.e. I show that the points in A but not in C are not in $Int(A)$: I take $X \in A, X \notin C$, I show that $X \notin Int(A)$ I construct a sequence (X_n) with $X_n \notin A$ but $X_n \to X$.
3. I think that $Adh(A) = B$ (it is necessary that $A \subset B$)
   (a) I show that B is closed (easy)
   $$\text{therefore } Adh(A) \subset B$$
   (b) We show that $B \subset Adh(A)$: We fix $X \in B$, we look for a sequence (X_n) with $X_n \in A$ and $X_n \xrightarrow[n \to \infty]{} X$. <u>We only look at the</u> $X \in B, X \notin A$

Example 2.47.

$$A = \{(x, y) \in \mathbb{R}^2 \mid 2x + 3y \leq 4, x \neq y\}$$

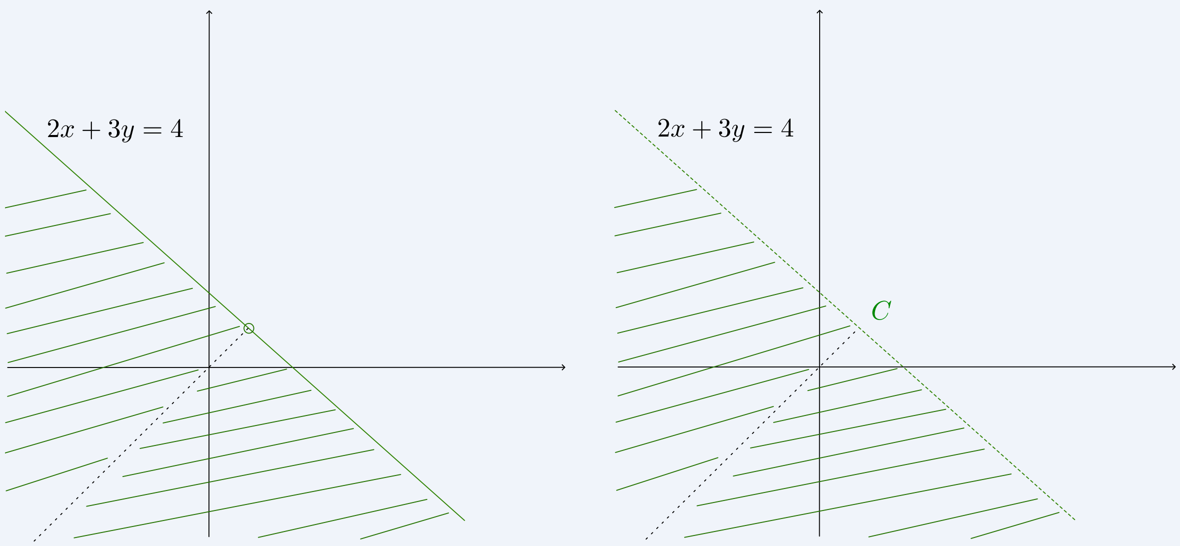

Figure 2.10: Example of the interior

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- I guess that $Int(A) = C = \{(x, y) \mid 2x + 3y < 4, x \neq y\}$
- Convect: $\{(x, y) \mid 2x + 3y < 4, x < y\} \cup \{(x, y) \mid 2x + 3y < 4, x > y\}$

I construct a sequence (X_n) with $X_n \notin A$ but $X_n \to X$. Let $X \in A, X \notin C$, $X = (x, y)$ therefore: $2x + 3y = 4 \quad x \neq y$

$$X_n = (x, y + \frac{1}{n})$$

$$2x_n + 3y_n = 2x + 3y + \frac{3}{n} = 4 + \frac{3}{n} > 4$$

$$X_n \notin A \text{ but } X_n \to X$$

Example 2.48.

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = x^{-1}\}$$

$Int(A) = \emptyset$? $C = \emptyset$

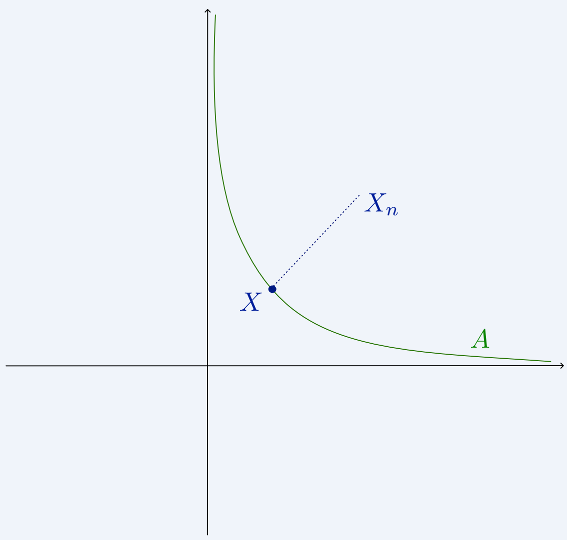

Figure 2.11: Example of the interior of the hyperbola

$\emptyset$ open, therefore $C \subset Int(A)$
Let $X \in A \quad X \notin C$, therefore $X \in A$.

$$X_n := (x, y + \frac{1}{n}) \quad X_n \notin A$$

$$x_n y_n = xy + \frac{x}{n} = 1 + \frac{x}{n} \neq 1$$

$$X_n \xrightarrow[n \to \infty]{} X \text{ therefore } X \notin Int(A)$$

$$Int(A) = \emptyset$$

Example 2.49.

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y = x^{-1}\}$$

$Adh(A) =$?

I think that $Adh(A) = A$ ($B = A$). It is sufficient to show that A <u>is closed</u>.

$$x > 0 \quad y \leq \frac{1}{x} \quad y \geq \frac{1}{x}$$

If $X_n = (x_n, y_n)$ $\quad X_n \in A$ and $X_n \to X$, then $X \in A$

$$X = (x, y) \quad \substack{x_n \to x \\ y_n \to y} \quad \substack{x_n \to x \\ \frac{1}{x_n} \to y} \quad (x_n > 0)$$

so $x > 0$ and $y = \frac{1}{x}$ so $X \in A$

A is closed

Example 2.50.

$$A = \{(x, y) \in \mathbb{R}^2 \mid 2x + 3y \leq 4, x \neq y\}$$

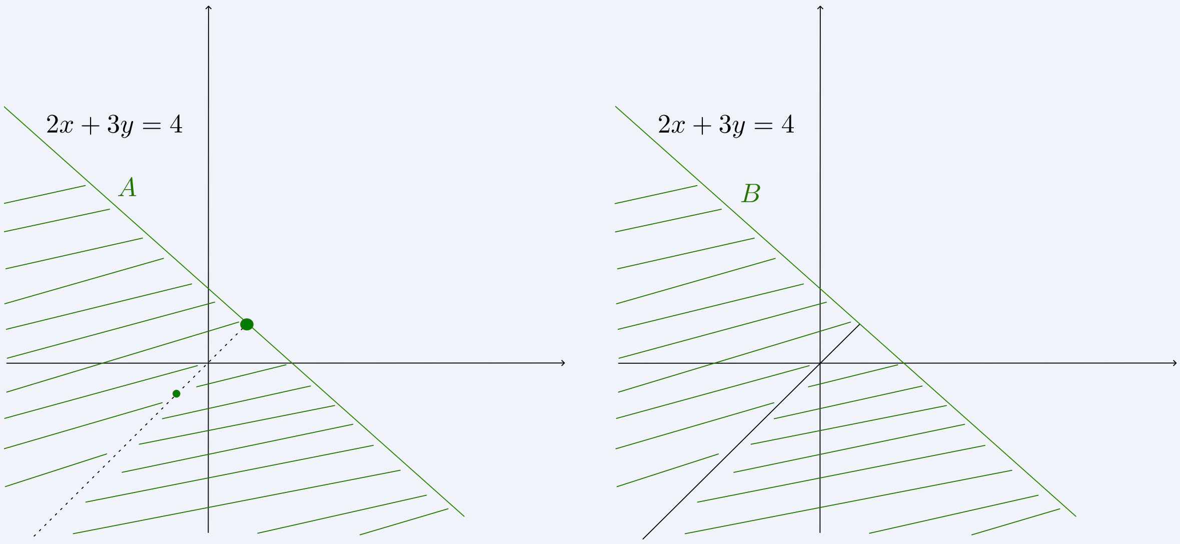

Figure 2.12: example-adherence

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1. B is closed (easy), therefore $Adh(A) \subset B$
2. Let $X \in B$. We show that $X \in Adh(A)$ (we look for $X_n \in A$ with $X_n \to X$)
   I'm just looking at $X \in B, X \notin A$

$$X_n = (x_n, y_n) \in A \quad x_n \to x \text{ and } y_n \to y$$

$$x_n = x + \frac{1}{n}, y_n = y = x$$

$$X_n \to X \text{ and } 2x_n + 3y_n = 2x + 3y - \frac{2}{n} \leq 4 and x_n \neq y_n$$

therefore $X_n \in A$

Example 2.51.

$$A = \{(x, y) \mid |x| \leq 1, |y| < 1\}$$

$$Int(A) = \{(x, y) \mid |x| < 1, |y| < 1\}$$

$$Adh(A) = \{(x,y) \mid |x| \leq 1, |y| \leq 1\}$$

Example 2.52.

$$A = \{(x,y) \mid x > 0, y = \sin(\frac{1}{n})\}$$

$Adh(A) = A \cup \{(0,y) \mid -1 \leq y \leq 1\}$ $Int(A) =$

fdsf fds fds

2.12 Compactness

Definition 2.53. Let $F \subset E$. An open cover of F is a collection $(U_i)_{i \in I}$ where U_i are open and $F \subset \cup_{i \in I} U_i$ ("the U_i cover F")

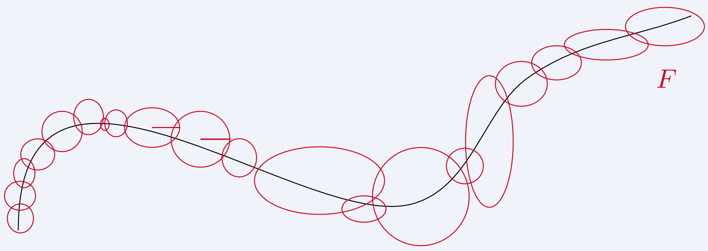

Figure 2.13: open-cover

Example 2.54.
- $U_x = B(x, \frac{1}{2})$
- $\bigcup_{x \in F} U_x$ contains F
- $(U_x)_{x \in F}$ open cover of F

Definition 2.55. $K \subset E$ is compact if from every open cover $(U_i)_{i \in I}$ of F we can extract a finite subcover: I can choose $i_1, \ldots, i_n \in I$ such that

$$F \subset U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n}$$

Property. A finite set is compact.

$$F = \{a_1, \ldots, a_p\} \quad a_j \in E$$

$(U_i)_{i \in I}$ covers F. I choose a_j (point of F), there exists an $i \in I$ denoted $i(j)$ such that

$$a_j \in U_{i(j)} \quad F \subset U_{i(1)} \cup \ldots \cup U_{i(p)}$$

Theorem 2.56. Characterization using sequences.

$K \subset E$ is compact iff every sequence of elements from K has a subsequence that converges to an element of K.

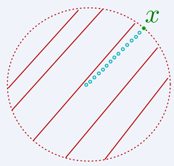

Figure 2.14: Compactness with sequences

Example 2.57.

- $E = \mathbb{R}^2$
- $F = B(x_0, r)$ not compact
- $x_n \in F$, $x_n \to x$, $x \notin F$
- if $y_n = x_{\phi(n)}$, $y_n \to x$ but $x \notin F$

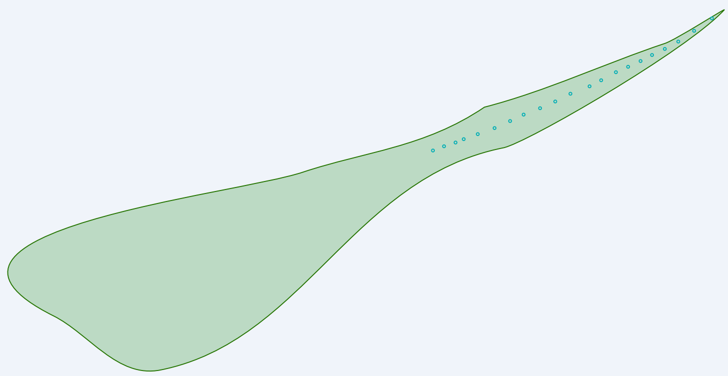

Figure 2.15: sequence-without-convergent-subsequence

Example 2.58.

$$F = \{(x, y) : x \geq 0, -\frac{1}{x} \leq y \leq \frac{1}{x}\}$$

$u_n = (n, 0)$ (u_n) sequence in F without convergent subsequence.

Proposition 2.59. 1. K compact $\Rightarrow$ K closed and bounded. (converse is false in general!)

2. If K compact and F closed, then $K \cap F$ is compact.

3. If K compact, every Cauchy sequence in K converges in K

Proof. 1. Let K be compact. K is closed if (u_n) is a sequence in K that converges to u, then $u \in K$.

clear: (u_n) has a subsequence $v_n = u_{\phi(n)}$ with $v_n \to v \in K$, $u_n \to u$, so $v_n \to u \Rightarrow u = v \Rightarrow u \in K$

K is bounded:

Let $U_x = \bigcup_{x \in K} B(x, 1)$ be an open cover of K. Now K is compact, so there exist $x_1, \ldots, x_n \in K$, such that $K \subset \bigcup_{i=1,\ldots,n} B(x_i, 1)$, so K is bounded.

2. K compact and F closed. (u_n) a sequence in $K \cap F$. $u_n \in K$. $\exists$ subsequence $v_n = u_{\phi(n)}$ with $v_n \to x \in K$. $v_n \in F, v_n \to x$, F closed so $x \in F$, $x \in K \cap F$.

3. Let (u_n) be a Cauchy sequence in K. (u_n) has a subsequence $v_n = u_{\phi(n)}$ that converges to $x \in K$. $u_n \to x \in K$

□

2.12.1 Compactness in $\mathbb{R}^n$ with the usual distance

Theorem 2.60. (Borel-Lebesgue)
in $\mathbb{R}^n$ with the usual distance K is compact iff K is closed and bounded

Proposition 2.61. Closed balls $B_f(x_0, r)$ are compact in $\mathbb{R}^n$.

- Implies the theorem: Let K be closed and bounded. K bounded, therefore $K \subset B_f(0, r)$ with r large, therefore $K = K \cap B_f(0, r)$. Therefore K is compact.

Proof. of prop. 2.61

1. $n = 1$. To show: $[a, b]$ is compact.

   Let $(U_i)_{i \in I}$ be an open cover of $[a, b]$. Let F: the $x \in [a, b]$ such that $[a, x]$ is covered by a finite number of U_i.

   Goal: to show that $b \in F$! (if $x \in F$, and $x' \leq x$ $x' \in F$)

   (a) $F \neq \emptyset$: $a \in F$ $[a, a] = \{a\}$

   (b) $c = sup(F)$. We show that $c = b$

   Suppose that $c < b$.

   - c belongs to one of the U_i denoted U_{i_0}
   - U_{i_0} is open, $c \in U_{i_0}$ so $\exists \delta_0 > 0$ such that $]c - \delta_0, c + \delta_0[\subset U_{i_0}$
   - $c = sup(F)$: $\forall \delta > 0$, $\exists x_\delta \in F$ with $c - \delta < x_\delta \leq c$

   $$\delta = \delta_{0,2} \quad \exists x_{\delta_0} \in F, c - \delta_{0,2} < x_{\delta_0}$$

   $[a, x_{\delta_0}]$ covered by $U_{i_1} \cup \ldots \cup U_{i_n}$ and $]c - \delta_0, c + \delta_0[\subset U_{i_0}$ so $[a, c + \delta_{0,2}]$ is covered by $U_{i_0} \cup U_{i_1} \cup \ldots \cup U_{i_n}$, so $c + \delta_{0,2} \in F$ contradicts that $c = sup(F)$. Thus $c = b$.
   F is $[a, b[$ or $[a, b]$. $b \in F$ $\exists U_{i_1}, \ldots, U_{i_n}$ such that $[a, b] \subset U_{i_1} \cup \ldots \cup U_{i_n}$, $[a, b]$ is compact.

□

2.13 Limits and continuity

2.13.1 Limits

I take $(E_1, d_1), (E_2, d_2)$ two metric spaces and $F : E_1 \to E_2$. $x_0 \in E_1, l \in E_2$.

Definition 2.62. .

1. Limit:
$$\lim_{x \to x_0} F(x) = l$$
if $\forall \varepsilon > 0, \exists \delta > 0$ such that if $d_1(x_0, x) < \delta$ then $d_2(l, F(x)) < \varepsilon$
2. F continuous at x_0 if $\lim_{x \to x_0} F(x) = F(x_0)$
3. F is continuous (on E) if it is continuous at every x_0 in E

Proposition 2.63. The following properties are equivalent:

1. $F : (E_1, d_1) \to (E_2, d_2)$ is continuous.
2. $\forall U_2 \subset E_2$ open, $F^{-1}(U_2)$ is open in E_1.
3. $\forall F_2 \subset E_2$ closed, $F^{-1}(F_2) \subset E_1$ is closed.
4. $\forall (x_n)$ sequence in E_1 with $\lim_{n\to\infty} x_n = x$ we have:
$$\lim_{n \to \infty} F(x_n) = F(x)$$

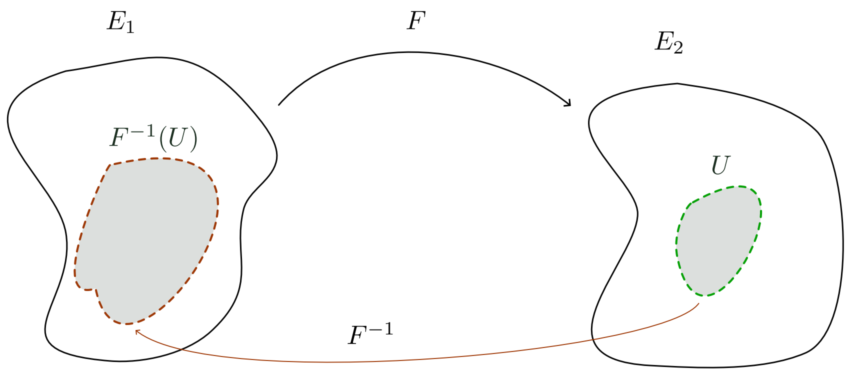

Figure 2.16: topological continuity

Example 2.64.
$$U = \{(x,y) \in \mathbb{R}^2 : x\sin(y) - e^x > 1\}$$
$$\begin{aligned} F : \mathbb{R}^2 &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto F((x,y)) = x\sin(y) - e^x \end{aligned}$$
obviously continuous.
$$U = F^{-1}(\underbrace{]1, +\infty[}_{\text{open in } \mathbb{R}})$$

Proof. $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 4 \Rightarrow 1$

$1 \Rightarrow 2$: Hyp: F continuous and $U_2 \subset E_2$ is open.
Conclusion: $U_1 = F^{-1}(U_2)$ is open?
I fix $x_0 \in U_1$ $(F(x_0) \in U_2)$.

1. U_2 open $\Rightarrow \exists \varepsilon_0 > 0$ s.t. $B_2(F(x_0), \varepsilon_0) \subset U_2$

2. F continuous at x_0:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } d_1(x_0, x) < \delta \Rightarrow d_2(F(x_0), F(x)) < \varepsilon$$

$$x \in B_1(x_0, \delta) \Rightarrow F(x) \in B_2(F(x_0), \varepsilon)$$

$\delta_0 =$ the δ that works for ε_0

$$x \in B_1(x_0, \delta_0) \Rightarrow F(x) \in B_2(F(x_0), \varepsilon_0)$$

Therefore $B_1(x_0, \delta_0) \subset F^{-1}(U_2)$. Therefore $F^{-1}(U_2)$ open.

$2 \Rightarrow 3$: : $F^{-1}(U_2)^c = F^{-1}(U_2^c)$

□

Example 2.65. result of this proposition. Let's take the function: $f(x) = x^2$. $f^{-1}(]4,9[) = \{x \in \mathbb{R} \mid 4 < x^2 < 9\} =]-3,-2[\cup]2,3[$. In other words, the continuity of f (obvious) implies that if $U =]4,9[$ is open, then $f^{-1}(U)$ is also open.

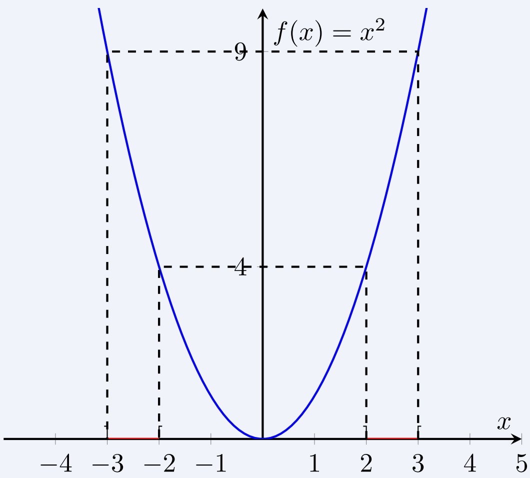

Figure 2.17: Example with $f(x) = x^2$

CHAPTER 3

Functions of several variables

3.1 Introduction

Framework: $\mathbb{R}^n, \mathbb{R}^p$ $D \subset \mathbb{R}^n$

$$F : D \to \mathbb{R}^p$$

on $\mathbb{R}^n$, $\mathbb{R}^p$ usual distances, on D the distance inherited from $\mathbb{R}^n$.
with Cartesian coordinates

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), F_2(x_1, \dots, x_n), \dots, F_p(x_1, \dots, x_n))$$

where $F_i : D \to \mathbb{R}$

$$F : D \to \mathbb{R}^p \text{ continuous}$$

we know:

Lemma 3.1.

$$F : D \to \mathbb{R}^p \text{ continuous if and only if:}$$

each $F_i : D \to \mathbb{R}$ is continuous

Proof. $Y_n = (Y_{1,n}, \dots, Y_{p,n})$ sequence of $\mathbb{R}^p$. $Y_n \to Y$ if and only if $Y_{i,n} \to Y_i$ $(1 \leq i \leq p)$ □

Proposition 3.2. Let $f, g : D \to \mathbb{R}$ be continuous.

- $f + g, f \times g$ are continuous on D
- if $g(X) \neq 0$, $\forall X \in D$, $\frac{f}{g}$ is continuous on D
- if $f(D) \subset I$ interval and $\phi : I \to \mathbb{R}$ is continuous, then $\phi \circ f : D \to \mathbb{R}$ is continuous.
- $$P : X \to \sum_{\alpha_1 + \ldots + \alpha_n \leq d} a_{\alpha_1, \dots, \alpha_n} x^{\alpha_1} \dots x^{\alpha_n}$$

  $a_{\alpha_1, \dots, \alpha_n} \in \mathbb{R}, d =$ degree of P.
  $P : \mathbb{R}^n \to \mathbb{R}$ is continuous.

3.2 How to show that a set is open or closed

According to the proposition 2.63, if $f : D \to Q$ is continuous and $K \subset Q$ open and $K_f \subset Q$ closed, then:

- $f^{-1}(K)$ is also open
- $f^{-1}(K_f)$ is also closed

This allows us to simplify the proofs that a set is closed or open. Here are some examples:

Example 3.3.

$$D = \{(x_1, x_2, x_3) : x_1^2 + 2x_2x_3^2 < 2, \sin(x_1x_2) > 0\}$$
$$D = D_1 \cap D_2$$

$$D_1 = f_1^{-1}(]-\infty, 2[) \qquad f_1(x) = x_1^2 + 2x_2x_3^2$$
$$D_2 = f_2^{-1}(]0, +\infty[) \qquad f_2(x) = \sin(x_1x_2)$$

D_1, D_2 are open, therefore D open.

Example 3.4.

$$D = \{(x_1, x_2) : \frac{e^{x_1 - 2x_2^2}}{x_1^2 + 3x_2^4} \geq 1\}$$

$$D = f^{-1}([1, +\infty[) \qquad f(x) = \frac{e^{x_1 - 2x_2^2}}{x_1^2 + 3x_2^4}$$

$[1, +\infty[$ is closed in $\mathbb{R}$, then D is also closed because f is continuous on $[1, +\infty[$

3.3 Connection with compactness

Theorem 3.5. Let $F : \mathbb{R}^n \to \mathbb{R}^p$ be continuous and $K \subset \mathbb{R}^n$ compact. Then, $F(K)$ is compact in $\mathbb{R}^p$

Remark 3.6. We can replace $\mathbb{R}^n, \mathbb{R}^p$ by E, F metric spaces.

Remark 3.7. U open, f continuous $\not\Rightarrow f(U)$ open:

Example 3.8.

$$f(]0, 1[) = [-1, 1]$$
$$f(x) = \sin(2\pi x)$$

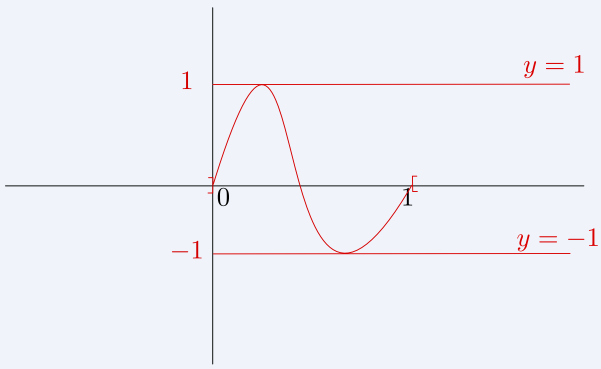

Figure 3.1: Example that an image of the open set is not open

Example 3.9.

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{R} \\ x &\longmapsto f(x) = \arctan x. \end{aligned}$$

$$f(\underbrace{]-\frac{\pi}{2}, \frac{\pi}{2}[}_{\text{not compact}}) = \underbrace{\mathbb{R}}_{\text{not compact}}$$

Proof. Let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $F(K)$. We have: $v_n = F(u_n)$ where $u_n \in K$. $(u_n)_{n\in\mathbb{N}}$ sequence in K, K compact, therefore: $\exists$ subsequence $(u_{\phi(n)})_{n\in\mathbb{N}}$ with

$$u_{\phi_n} \xrightarrow[n\to+\infty]{} u \in K$$

F continuous: therefore $F(u_{\phi(n)}) = v_{\phi(n)} \to F(u) \in K$. (v_n) has a subsequence $(v_{\phi(n)})$ that converges to $F(u) \in F(K)$, therefore $F(K)$ compact! □

Theorem 3.10. Let $F : \mathbb{R}^n \to \mathbb{R}$ continuous and $K \subset \mathbb{R}^n$ compact. Then f is bounded on K and attains its bounds. That is, $Q := f(K)$ is bounded and attains the bounds.

Proof. Weierstrass: $f : \mathbb{R} \to \mathbb{R}$ $K = [a, b]$.

I take (E, d) instead of $\mathbb{R}^n$. f bounded on K: $\exists c_1, c_2$ such that

$$c_1 \leq f(x) \leq c_2, \forall x \in K \Leftrightarrow f(K) \subset [c_1, c_2]$$

It is clear because $f(K)$ is compact in $\mathbb{R}$, therefore bounded.

$$m = \underset{x\in K}{inf} f(x) = inf\ f(K) \qquad\qquad M = \underset{x\in K}{sup} f(x) = sup\ f(K)$$

To show: $\exists x \in K$ such that $f(x) = m$ and $\exists x' \in K$ such that $f(x') = M$

$m = inf\ f(K)$, that means that

1. $f(K) \subset [m, +\infty[$ (m a lower bound for $f(K)$)
2. $\forall \varepsilon > 0, \exists y \in f(K)$ such that $y \leq m + \varepsilon$

$\varepsilon = \frac{1}{n}$ gives a sequence $y_n \in f(K)$ such that $y_n \to m$

$$y_n = f(x_n) \qquad\qquad x_n \in K$$

K compact: $\exists$ subsequence $x_{\phi(n)}$ such that

$$x_{\phi(n)} \xrightarrow[n\to\infty]{} x \in K$$

$f : E \to \mathbb{R}$ continuous, therefore

$$f(x_{\phi(n)}) = y_{\phi(n)} \to f(x)$$

But, $y_n \to m$, so $y_{\phi(n)} \to m$ and $y_{\phi(n)} \to f(x)$, so $m = f(x)$, m is attained.

To show that M is attained the proof is identical. □

3.4 Partial continuity (useless)

$$D \subset \mathbb{R}^n \quad f : D \to \mathbb{R} \text{ continuous} \quad D \text{ open}$$

Let $A = (a_1, \ldots, a_n) \in D$, there exist open intervals $I_1, \ldots, I_n$ with $a_i \in I_i$ such that $I_1 \times \ldots \times I_n \subset D$

I can set

$$f_i(t) = f(a_1, \dots, a_{i-1}, t, a_{i+1}, \dots, a_n) \qquad t \in I_i$$

Example 3.11.

$$n = 2 \quad f_1(t) = f(t, a_2) \quad f_2(t) = f(a_1, t)$$

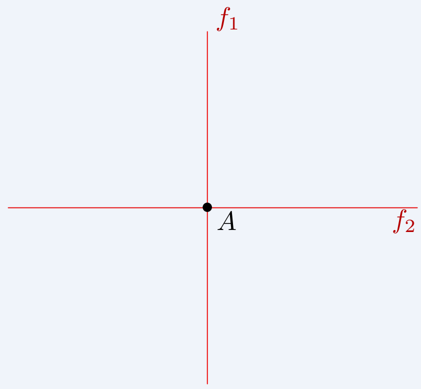

Figure 3.2: f is continuous at $A = (a_1, a_2)$

Definition 3.12. f is partially continuous at $A = (a_1, \dots, a_n)$ if the $f_i(t)$ are continuous at a_i $(1 \leq i \leq n)$

- continuity: $f(x_1, x_2) \xrightarrow[(x_1,x_2)\to(a_1,a_2)]{} f(a_1, a_1)$
- partial: $f(x_1, a_2) \xrightarrow[x_1 \to a_1]{} f(a_1, a_2)$ et $f(a_1, x_2) \xrightarrow[x_2 \to a_2]{} f(a_1, a_2)$
- Good notion: continuity implies partial continuity (converse false)

Example 3.13.

$$f(x_1, x_2) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2} \text{ if } (x_1, x_2) \neq (0,0) \\ 0 \text{ if } (x_1, x_2) = (0,0) \end{cases}$$

continuous on $\mathbb{R}^2 \setminus \{(0,0)\}$

- partially continuous at $(0,0)$

$$f(x_1, 0) = \begin{cases} 0 \text{ if } x_1 = 0 \\ 0 \text{ if } x_1 \neq 0 \end{cases}$$

$$f(0, x_2) = 0 \, \forall x_2$$

- not continuous at $(0,0)$:

$$x_1 = r\cos(\theta) \quad x_2 = r\sin(\theta)$$

$$f(r\cos(\theta), r\sin(\theta)) = \begin{cases} 0 \text{ if } r = 0 \\ \frac{r^2\cos(\theta)\sin(\theta)}{r^2} = \cos(\theta)\sin(\theta) \text{ if } r \neq 0 \end{cases}$$

$$\lim_{r\to 0} f(r\cos(\theta), r\sin(\theta)) = \cos(\theta)\sin(\theta) \neq 0 \text{ if } \theta \neq 0, \pi, \frac{\pi}{2}, \dots$$

CHAPTER 4

Differentiation of functions of several variables

4.1 Introduction

$n = 1$: how to define $f'(x_0)$?

1. $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$
2. LE: $f(x) = f(x_0) + a_1(x - x_0) + (x - x_0)\varepsilon(x)$ where $a_1 = f'(x_0)$

$$f : D \to R \quad D \text{ open} \quad X_0 \in D \quad D \subset \mathbb{R}^n$$

Definition 4.1. f is differentiable at X_0 in the direction $\vec{u}$ $(\neq \vec{0})$ if the function

$$\begin{aligned} g : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto g(t) = f(X_0 + t\vec{u}). \end{aligned}$$

is differentiable at $t = 0$

In other words, the directional derivative (in the direction of vector $\vec{u}$) is given by:

$$D_u f(X_0) = \lim_{t \to 0} \frac{f(X_0 + t\vec{u}) - f(X_0)}{t} \tag{4.1}$$

In the case $\mathbb{R}$ we had the definition of the derivative:

$$f'(x_0) = \lim_{t \to 0} \frac{f(x_0 + t) - f(x_0)}{t}$$

The direction was always the same (the x-axis), we can see this as taking a vector $u = (1)$ and using only the x-axis as the direction and we obtain eq. (4.1)

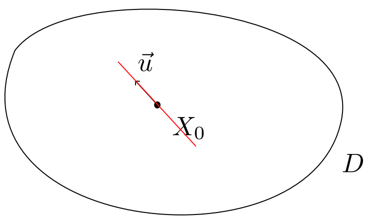

Figure 4.1: Directional derivative

$\vec{e_1}, \dots, \vec{e_n}$ canonical basis of $\mathbb{R}^n$, f has partial derivatives at X_0 if f is differentiable at X_0 in the directions $\vec{e_1}, \dots, \vec{e_n}$.

$$\frac{d}{dt} f(X_0 + t\vec{e_i}) \mid_{t=0}$$

denoted

$$\frac{\partial f}{\partial x_i}(X_0)$$

On the other hand, a function can be differentiable in<u>all directions</u>at a point but<u>not to be</u>continues at this point, here is

Example 4.2.

$$f(x_1, x_2) = \begin{cases} 1 \text{ if } x_2 = x_1^2 \text{ and } (x_1, x_2) \neq (0,0) \\ 0 \text{ otherwise} \end{cases}$$

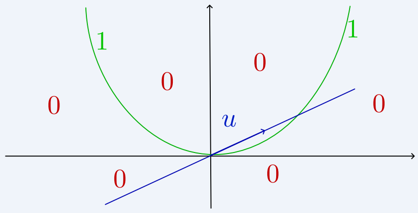

Figure 4.2: Example differentiable but not continuous

$$f((0,0) + t\vec{u}) = f(t\vec{u}) = 0$$

if $t \neq 0$ and t small, f is differentiable in all directions.

But, f is not continuous at $(0,0)$:

$$X_n = (\frac{1}{n}, \frac{1}{n^2}) \quad X_n \to (0,0)$$

$$\forall n, f(X_n) = 1 \quad f(X_n) \underset{n\to\infty}{\not\longrightarrow} f(0,0)$$

Definition 4.3. Let $D \subset \mathbb{R}^n$ be open and $X_0 \in D$, the function $f : D \to \mathbb{R}$ is **differentiable** at X_0 if there exists a vector $\vec{u} \in \mathbb{R}^n$ such that

$$f(X_0 + \vec{X}) = f(X_0) + \vec{u} \cdot \vec{X} + \|\vec{X}\|\varepsilon(\vec{X})$$

where $\lim_{\vec{X}\to\vec{0}} \varepsilon(\vec{X}) = 0$

Intuition. I propose to reflect on what this definition means. Let's recall what the derivative intuitively means in the case $\mathbb{R}^n = \mathbb{R}$ ($n = 1$). Intuitively, if we zoom in on the function we are deriving, it behaves and looks like a line. In the case $R^n = \mathbb{R}^2$, if we zoom in on the function, it looks like a plane. Indeed, that's the idea of the derivative: if we take a tiny step of an ant, the displacement is also small and uniform. By increasing n, the derivative gives scalars to construct a subspace of dimension $n - 1$ of the space $\mathbb{R}^n$.

Note. To show that a function is differentiable, it is sufficient to show that its partial derivatives are continuous.

4.2 First-order Taylor expansion

This representation of the derivative as a subspace when zooming is represented by the Taylor expansion of order 1. From the definition 4.3, this vector $\vec{u}$ is denoted $\vec{\nabla} f(X_0)$ (gradient of f at X_0)

Proposition 4.4. f differentiable at $X_0 \Rightarrow f$ differentiable in all directions at X_0, and then:

$$\vec{\nabla} f(X_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1} f(X_0) \\ \dots \\ \frac{\partial f}{\partial x_n} f(X_0) \end{pmatrix}$$

in the basis $\vec{e_1}, \dots, \vec{e_n}$

Proof. f is continuous at X_0 $|\vec{u} \cdot X| \leq |\vec{u}||X|$

1. continuity

$$\begin{aligned} |f(X_0 + X) - f(X_0)| &\leq |\vec{u} \cdot X| + \|X\||\varepsilon(X)| \\ &\leq \|X\| \left(\|\vec{u}\| + |\varepsilon(x)|\right) \leq c\|X\| \end{aligned}$$

therefore: $f(X_0 + X) \xrightarrow[X \to \vec{0}]{} f(X_0)$

2. .

$$\begin{aligned} g(t) = f(X_0 + t\vec{v}) &= f(X_0) + \vec{\nabla} f(X_0) \cdot t\vec{v} + \|t\vec{v}\| \cdot \varepsilon(t\vec{v}) \\ &= f(X_0) + t\vec{\nabla} f(X_0) \cdot \vec{v} + |t|\|\vec{v}\|\varepsilon_1(t) \\ &= f(X_0) + t\vec{\nabla} f(X_0) \cdot \vec{v} \end{aligned}$$

therefore:

$$\frac{d}{dt} f(X_0 + t\vec{v}) \mid_{t=0} = \vec{\nabla} f(X_0) \cdot \vec{v}$$

(take $\vec{v} = \vec{e_1}, \dots, \vec{e_n}$ for the coordinates of $\vec{\nabla} f(X_0)$)

∎

Definition 4.5.

$$D \subset \mathbb{R}^n \quad D \text{ open} \quad f : D \to \mathbb{R} \text{ is } \mathcal{C}^1 \text{ on } D$$

Let $D \subset \mathbb{R}^n$ be open, then the function $f : D \to \mathbb{R}$ is of class $\mathcal{C}^1$ on D if f is differentiable at every $X \in D$ and the function

$$\begin{aligned} : D &\longrightarrow \mathbb{R}^n \\ X &\longmapsto \vec{\nabla} f(X) \end{aligned}$$

is continuous.

Theorem 4.6. f of class $\mathcal{C}^1$ on D ssi f has continuous partial derivatives at every point of D.

Example 4.7.

$$f(X) = f(X_0) + \vec{\nabla} f(X_0) \cdot (X - X_0) + \|X - X_0\|\varepsilon(X - X_0)$$

linear

In $\mathbb{R}^3$: $f(x, y, z)$

$$S = \{(x, y, z) : f(x, y, z) = 0\}$$

S: surface in $\mathbb{R}^3$, $X_0 \in S$ tangent plane to S at X_0, plane equation:

$$f(X_0) + \vec{\nabla} f(X_0) \cdot X = 0$$

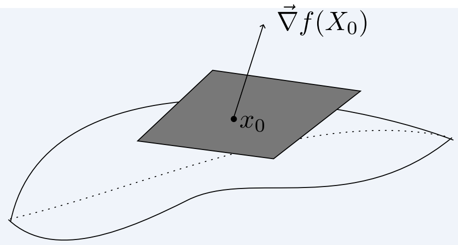

Figure 4.3: Example of a differentiable surface

4.3 Extrema and critical points

Definition 4.8. Extremum (local) of f is a minimum or a maximum (local) of f

- X_0 is a local maximum of f if: $\exists \delta > 0$ such that

$$\forall X \in D, f(X) \leq f(X_0) \text{ with } d(X, X_0) \leq \delta$$

- X_0 is a local minimum of f if: $\exists \delta > 0$ such that

$$\forall X \in D, f(X) \geq f(X_0) \text{ with } d(X, X_0) \leq \delta$$

Definition 4.9. Let $f : D \to \mathbb{R}$ and $X_0 \in D$, then if

$$\vec{\nabla} f(X_0) = \vec{0}$$

therefore X_0 is a **critical point**.

Intuition. The link between extremums and the critical point:

1. for the extremum to exist, there must be at least one critical point - it is a necessary <u>but not sufficient</u> criterion.
2. every local extremum is a critical point

Critical points facilitate the search for local extremums.

Theorem 4.10. Let $f : D \longrightarrow \mathbb{R}$ differentiable, D open and $X_0 \in D$ (otherwise, if D not open, it must be $X_0 \in \text{Int}(D)$) then:

$$X_0 \text{ local extremum } \Rightarrow X_0 \text{ critical point}$$

Example 4.11. Not every critical point is a local extremum

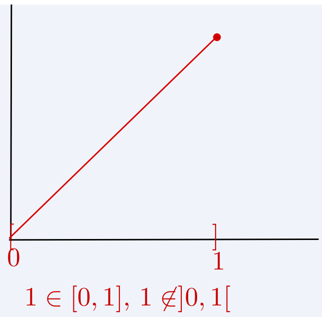

Figure 4.4: Critical point that is not a local extremum

4.4 Partial derivatives of order ≥ 2

Definition 4.12. Let D, then $f : D \to \mathbb{R}$ is $\mathcal{C}^k$ if $f : D \to \mathbb{R}$ is $\mathcal{C}^1$ and $\partial_{x_i} f : D \to \mathbb{R}$ are C^{k-1}

Definition 4.13. Let $\alpha = (\alpha_1, \dots, \alpha_n) \quad \alpha_i \in \mathbb{N}$. We define

$$\partial_x^\alpha f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \dots \cdot \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$$

is the notation for the higher-order derivative.

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1} f \stackrel{?}{=} \frac{\partial^2}{\partial x_1^2}\frac{\partial}{\partial x_2} f$$

Theorem 4.14. Schwarz Lemma
If $f \in \mathcal{C}^2(D)$ then

$$\frac{\partial^2 f}{\partial x_i \partial x_j}(X) = \frac{\partial^2 f}{\partial x_j \partial x_i}(X) \qquad \forall X \in D, \forall i, j$$

Example 4.15. where a function has higher-order partial derivatives but $\frac{\partial^2 f}{\partial x_i \partial x_j}(X) \neq \frac{\partial^2 f}{\partial x_j \partial x_i}(X)$

$$f(x_1, x_2) = \begin{cases} x_1 x_2 \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2} \text{ if } (x_1, x_2) \neq (0, 0) \\ 0 \text{ if } (x_1, x_2) = 0 \end{cases}$$

$$r^2 \sin(\theta)\cos(\theta)\cos(2\theta) = \frac{1}{4} r^2 \sin(4\theta)$$

We calculate $\frac{\partial^2 f}{\partial_{x_1} \partial_{x_2}}(0, 0)$? It is $\frac{\partial}{\partial x_1} g(x_1)$ at $x_1 = 0$ for $g(x_1) = \frac{\partial f}{\partial x_2}(x_1, x_2)|_{x_2=0}$. Calculation of $g(x_1)$:

1. if $x_1 \neq 0$ $\frac{\partial f}{\partial x_2}(x_1, x_2) = x_1 \frac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$, so if $x_1 \neq 0$ $\frac{\partial f}{\partial x_2}(x_1, 0) = x_1$
2. if $x_1 = 0$ $f(0, x_2) = 0$

Conclusion:

$$\frac{\partial f}{\partial x_2}(x_1, 0) = x_1 \quad \forall x_1$$

so:

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2} f(0, 0) = 1$$

$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}f(0,0) =?$. We see that, $f(x_2, x_1) = -f(x_1, x_2)$ so

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}f(0,0) = -\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}f(0,0) = -1$$

4.5 Taylor's Formula to Order 2

Definition 4.16. Let $f \in \mathcal{C}^2(D)$. Hessian matrix: $n \times n$ matrix

$$H_f(X_0) = \left[\frac{\partial^2}{\partial x_i \partial x_j}(X_0)\right] 1 \leq i, j \leq n$$

The lemma 4.14 tells us that $H_f(X_0)$ is symmetric if $f \in \mathcal{C}^2(D)$

Recall:

$$\vec{\nabla} f(X_0) = \begin{pmatrix} \frac{\partial f}{\partial x_1}(X_0) \\ \vdots \\ \frac{\partial f}{\partial x_n}(X_0) \end{pmatrix}$$

Theorem 4.17. Taylor of order 2
Let $f \in \mathcal{C}^2(D)$, $X_0 \in D$. Then

$$f(X_0 + \vec{X}) = f(X_0) + \vec{\nabla} f(X_0) \cdot \vec{X} + \frac{1}{2}\vec{X} \cdot H_f(X_0)\vec{X}$$

example in $\mathbb{R}^1$

$$f(x_0 + x) = f(x_0) + f'(x_0)x + \frac{1}{2}f''(x_0)x^2 + \dots$$

Intuition. So, the Hessian matrix is used to calculate the second-order derivative.

4.6 A reminder of linear algebra and the link with analysis

$$\vec{X} \cdot A\vec{X} = \sum_{1 \leq i,j \leq n} x_i a_{i,j} x_j$$

If $\vec{X} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ $A = [a_{i,j}]$ we have: $X \mapsto X \cdot AX$ to study. If $A = A^T, A \in \mathcal{M}_n(\mathbb{R})$

"A admits an orthonormal basis of eigenvectors"

There exists a basis $\vec{u_1}, \dots, \vec{u_n}$ of $\mathbb{R}^n$ with $\vec{u_i} \cdot \vec{u_j} = \delta_{i,j}$ (1 if $i = j$ and 0 otherwise) and real numbers $\lambda_1, \dots, \lambda_n (\lambda_i = \lambda_j$ possible) such that

$$A\vec{u_i} = \lambda_i \vec{u_i}$$

$$\vec{X} = \sum_{j=1}^{n} y_j \vec{u_j}$$

$$\vec{X} \cdot \vec{u_i} = \sum_{j=1}^{n} y_j \vec{u_j} \vec{u_i} = y_i$$

$$\|\vec{X}\|^2 = \vec{X}\cdot\vec{X} = \left(\sum_{j=1}^{n} y_j\vec{u_j}\right)\cdot\left(\sum_{i=1}^{n} y_i\vec{u_i}\right)$$
$$= \sum_{j=1}^{n}\sum_{i=1}^{n} y_j y_i \vec{u_j}\cdot\vec{u_i}$$
$$= \sum_{j=1}^{n} y_j^2$$

$$A\vec{X} = A\sum_{j=1}^{n} y_j\vec{u_j} = \sum_{j=1}^{n} y_j A\vec{u_j} = \sum_{j=1}^{n}\lambda_j y_j\vec{u_j}$$

$$\vec{X}\cdot A\vec{X} = \sum_{i=1}^{n}\lambda_i y_i^2$$

1. if $\lambda_i > 0$ $(1 \le i \le n)$
$$C = \min \lambda_i > 0$$
$$X\cdot AX \ge C\sum_{i=1}^{n} y_i^2 = C\|X\|^2$$
2. if $\lambda_i < 0$ $(1 \le i \le n)$
$$-C = \max \lambda_i < 0$$
$$X\cdot AX \le -C\|X\|^2$$

Example 4.18. $n = 2$
$$f(y_1, y_2) = -y_1^2 + 3y_2^2$$
$$\lambda_1 = -1 \qquad \lambda_2 = 3$$
$$f(y_1, 0) < f(0,0) < f(0, y_2)$$

4.7 Nature of critical points

Theorem 4.19. (Nature of critical points)
Let $f \in \mathcal{C}^2(D)$, $X_0 \in D$, D open and $\vec{\nabla} f(X_0) = \vec{0}$

1. if all eigenvalues of $H_f(X_0)$ are > 0 (respectively < 0) X_0 is a local minimum (respectively maximum) local.
2. if all eigenvalues of $H_f(X_0)$ are non-zero but not of the same sign, X_0 is not a local extremum: X_0 is a saddle point (a pass).
3. if 0 is an eigenvalue of $H_f(X_0)$, no conclusion, (X_0 degenerate critical point) that is we cannot conclude anything

Proof. of the theorem 4.19
$$f(X_0 + X) - f(X_0) = \frac{1}{2}X\cdot H_f(X_0)X + \|X\|^2\varepsilon(X)$$
.

1. if $\lambda_i > 0$ $\frac{1}{2}X\cdot H_f(X_0)X \ge C\|X\|^2$ $C > 0$
$$f(X_0+X) - f(X_0) \ge \|X\|^2(C + \varepsilon(X)) \ge \frac{C}{2}\|X\|^2 \text{ if } \|X\| \text{ small enough}$$

$$\Rightarrow X_0 \text{ local minimum}$$

2. if $\lambda_1 < 0$ and $\lambda_2 > 0$

$$H_f(X_0)\vec{u_i} = \lambda_i \vec{u_i}$$

$$f(X_0 + t\vec{u_i}) = f(X_0) + \frac{1}{2}\lambda_i t^2 + t^2\varepsilon(t)$$

$$\varepsilon(t\vec{u_i}) = \varepsilon(t)$$

$$f(X_0 + t\vec{u_i}) - f(X_0) = t^2(\frac{1}{2}\lambda_i + \varepsilon(t))$$

if $i = 1 < 0$ $|t|$ small, $i = 2 > 0$ $|t|$ small, then X_0 is not a local extremum

□

Example 4.20.

$$f(x, y) = \frac{1}{2}(x^2 - y^2)$$

$$H_f(0, 0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$I_f = \{(x, y, z) : z = \frac{1}{2}(x^2 - y^2)\}$$

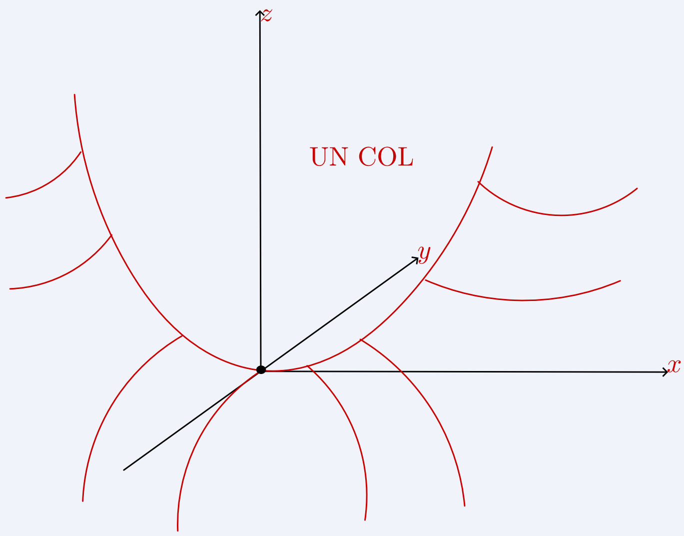

Figure 4.5: Example of saddle point.

The red lines represent the partial derivatives and we can clearly see that some are increasing and others decreasing, so this point is neither the minimum nor the maximum

Example 4.21. $n = 2$

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix}$$

$(a_{1,2} = a_{2,1})$

Eigenvalues: roots of the characteristic polynomial:

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} a_{1,1} - \lambda & a_{1,2} \\ a_{2,1} & a_{2,2} - \lambda \end{vmatrix} = (\lambda - a_{1,1})(\lambda - a_{2,2}) - a_{1,2}a_{2,1}$$

$$\lambda^2 - (a_{1,1} + a_{2,2})\lambda + a_{1,1}a_{2,2} - a_{2,1}a_{1,2}$$

$$a_{1,1} + a_{2,2} = Tr(A)$$

$$a_{1,1}a_{2,2} - a_{2,1}a_{1,2} = \det(A)$$

$$x^2 - Sx + P = x^2 - (\lambda_1 + \lambda_2)x + \lambda_1\lambda_2$$

$$\det(A) = \text{ product of eigenvalues}$$
$$Tr(A) = \text{ sum of eigenvalues}$$

$$A = H_f(X_0)$$

1. if $\det(A) < 0$, X_0 saddle point
2. if $\det(A) > 0$
   (a) $Tr(A) > 0$, X_0 minimum
   (b) $Tr(A) < 0$, X_0 maximum
3. $\det(A) = 0$, X_0 degenerate critical point

4.8 The Chain Rule

Definition 4.22. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function is differentiable and functions $g_1 : \mathbb{R} \to \mathbb{R}$, ..., $g_n : \mathbb{R} \to \mathbb{R}$ be differentiable and continuous functions and

$$\begin{aligned} h : \mathbb{R} &\longrightarrow \mathbb{R} \\ t &\longmapsto h(t) = f(g_1(t), g_2(t), \dots, g_n(t)) \end{aligned}$$

then

$$h'(t) = \frac{\partial g_1}{\partial h} g_1'(t) + \frac{\partial g_2}{\partial h} g_2'(t) + \dots + \frac{\partial g_n}{\partial h} g_n'(t)$$

Definition 4.23. Let $f : \mathbb{R}^n \to \mathbb{R}$ a continuous function is differentiable and functions $g_1 : \mathbb{R}^p \to \mathbb{R}$, ..., $g_n : \mathbb{R}^p \to \mathbb{R}$ differentiable functions i.e.

$$\begin{aligned} \forall i \in \{1, \dots, n\}, \quad g_i : \mathbb{R}^p &\longrightarrow \mathbb{R} \\ (t_1, \dots, t_n) &\longmapsto g_i(t_1, \dots, t_n) \end{aligned}$$

and

$$\begin{aligned} h : \mathbb{R}^n &\longrightarrow \mathbb{R} \\ (x_1, \dots, x_n) &\longmapsto h(g_1(t_1, \dots, t_p), \dots, g_n(t_1, \dots, t_p)). \end{aligned}$$

therefore

$$\frac{\partial h}{\partial t_i} = \frac{\partial h}{\partial x_1}\frac{\partial g_1}{\partial t_i} + \dots + \frac{\partial h}{\partial x_n}\frac{\partial g_n}{\partial t_i}$$

CHAPTER 5

Normed Vector Spaces

5.1 Introduction

Definition 5.1. Let E be a $\mathbb{K}$-vector space and $\lambda \in \mathbb{R}$, the **norm** on E is a map $N : E \to \mathbb{R}_+$ with:

1. $N(\lambda u) = |\lambda| N(u) \quad u \in E$
2. $N(u+v) \leq N(u) + N(v)$
3. $N(u) = 0 \Leftrightarrow u = 0_E$

semi-norm: 1 and 2 only.

We can interpret 2 as:

$$|N(u) - N(v)| \leq N(u-v)$$

Proposition 5.2. **Induced norm**: If $F \subset E$ a vector subspace, I restrict N to F, then (F, N) is a normed vector space.

Example 5.3. $E = \mathbb{K}^n$ with $x = (x_1, \ldots, x_n) \in E$

- $\|x\|_1 = \sum_{i=1}^n |x_i|$
- $\|x\|_2 = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$
- $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$
- $\|x\|_p = (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$ with $1 \leq p < \infty$

Proposition 5.4. The triangle inequality for $p > 2$ is called **Minkowski's inequality**:

$$\left(\sum_{i=1}^n |x_i + y_i|^p\right)^{\frac{1}{p}} \leq \left(\sum_{i=1}^n |x_i|^p\right)^{\frac{1}{p}} + \left(\sum_{i=1}^n |y_i|^p\right)^{\frac{1}{p}}$$

Definition 5.5. Let U be a set and $E = \{f : U \to \mathbb{K} \text{ bounded}\}$

$$\|f\|_\infty = \sup_{x \in U} |f(x)| \text{ norm on E}$$

Definition 5.6. $R([a,b],\mathbb{K}) = \{$ the $f : [a,b] \to \mathbb{K}$ Riemann integrable[a] $\}$

[a]The function is Riemann integrable (not necessarily continuous) if one can calculate the area using integration by Riemann sums. Then, if f is discontinuous, it is Riemann integrable if the discontinuity is negligible.

Example 5.7.

$$\|f\|_p = \left(\int_a^b |f(x)|^p \, dx\right)^{\frac{1}{p}} \quad \text{with } 1 \leq p < \infty$$

$\|.\|_p$ is a semi-norm on $R([a,b],\mathbb{K})$ (Minkowski's inequality). $\|f\|_p = 0$ does not imply that $f = 0$ (e.g.: $[a,b] = [-1,1]$, $f(x) = x$, $p = 3$).

$$\|u+v\|_p \leq \|u\|_p + \|v\|_p$$

On $E = \mathcal{C}([a,b],\mathbb{K})$, $\|.\|_p$ is a norm: if $f : [a,b] \to \mathbb{K}$ continuous and $\int_a^b |f(x)|^p \, dx = 0$ then $f(x) = 0 \forall x \in [a,b]$

Example 5.8. $E = \mathbb{K}^{\mathbb{N}}$ a set of sequences u with values in $\mathbb{K}$

$$u = (u_1, u_2, \ldots, u_n, \ldots)$$

for $1 \leq p < \infty$

$$l^p(\mathbb{N},\mathbb{K}) = \{(u_n) : \sum_{n\in\mathbb{N}} |u_n|^p \text{ is convergent } \}$$

$$\|u\|_p = \left(\sum_{n=0}^{\infty} |u_n|^p\right)^{\frac{1}{p}}$$

is a norm on $l^p(\mathbb{N},\mathbb{K})$

$$p = \infty \quad l^\infty(\mathbb{N},\mathbb{K}) = \{u \text{ bounded } \}$$

$$\|u\|_\infty = \sup_{n\in\mathbb{N}} |u_n|$$

5.2 Topology of normed vector spaces

Proposition 5.9. Let $(E, \|.\|)$ be a normed vector space with

$$d(u,v) = \|u - v\|$$

a distance on E (induced by $\|.\|$), then (E,d) is a metric space.

Definition 5.10. A complete normed vector space is called **a Banach space**.

Finite-dimensional case:

1. Every finite-dimensional normed vector space is complete (recall: proposition 2.43) (see below)
2. If E of finite dimension:

$$K \text{ compact} \Leftrightarrow K \text{ closed and bounded}$$

Lemma 5.11.

$$(\mathcal{C}([0,1],\mathbb{R}), \|.\|_1)$$

is not complete.

Proof. We construct a sequence of continuous functions $(f_n)_{n\in\mathbb{N}}$ on $[0,1]$ that converges in the norm $\|\cdot\|_1$ to a discontinuous function f. This will show that the limit of this sequence in the norm $\|\cdot\|_1$ does not belong to $\mathcal{C}([0,1],\mathbb{R})$, therefore this space is not complete.

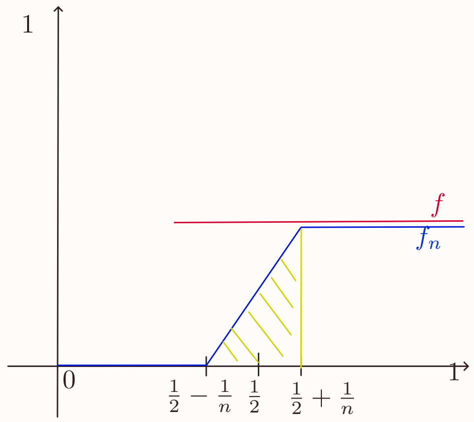

Figure 5.1: Lemma with a non-complete space

Definition of the sequence (f_n) : for each $n \in \mathbb{N}$, we define $f_n : [0,1] \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq \frac{1}{2} - \frac{1}{2n}, \\ 2n\left(x - \frac{1}{2} + \frac{1}{2n}\right) & \text{if } \frac{1}{2} - \frac{1}{2n} < x < \frac{1}{2} + \frac{1}{2n}, \\ 1 & \text{if } x \geq \frac{1}{2} + \frac{1}{2n}. \end{cases}$$

Each f_n is continuous on $[0,1]$ because it is piecewise affine with continuous connections.

Definition of the limit function : let's set

$$f(x) = \begin{cases} 0 & \text{if } x < \frac{1}{2}, \\ 1 & \text{if } x > \frac{1}{2}, \\ \text{any value} & \text{if } x = \frac{1}{2}. \end{cases}$$

Then f is **discontinuous** at $x = \frac{1}{2}$, so $f \notin \mathcal{C}([0,1],\mathbb{R})$.

Convergence of (f_n) to f in $\|\cdot\|_1$:
We have

$$\|f_n - f\|_1 = \int_0^1 |f_n(x) - f(x)|\, dx.$$

But $f_n(x) = f(x)$ except on the interval $\left[\frac{1}{2} - \frac{1}{2n}, \frac{1}{2} + \frac{1}{2n}\right]$ of length $\frac{1}{n}$, and on this interval, $|f_n(x) - f(x)| \leq 1$, so :

$$\|f_n - f\|_1 \leq \int_{\frac{1}{2}-\frac{1}{2n}}^{\frac{1}{2}+\frac{1}{2n}} 1\, dx = \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Thus, $f_n \to f$ in the norm $\|\cdot\|_1$.

Consequence : the sequence (f_n) is Cauchy in $(\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_1)$, because :

$$\|f_n - f_p\|_1 \leq \|f_n - f\|_1 + \|f - f_p\|_1 \leq \frac{1}{n} + \frac{1}{p} \xrightarrow[n,p\to\infty]{} 0.$$

However, the limit f is not continuous, so $f \notin \mathcal{C}([0,1],\mathbb{R})$.

Conclusion : There exists a Cauchy sequence in $(\mathcal{C}([0,1],\mathbb{R}), \|\cdot\|_1)$ that does not converge in this space. Therefore, this space is not complete.

□

Lemma 5.12. In $E = l^1(\mathbb{N},\mathbb{R})$ equipped with

$$\|u\|_1 = \sum_{n=0}^{\infty} |u_n|$$

$B_f(0,1)$ is not compact.

Proof. We construct a sequence of elements from $B_f(0,1)$ without a convergent subsequence.

$$u \in E \quad u : \mathbb{N} \to \mathbb{R}$$

I denote $u(p)$ instead of u_p sequence in E denoted (u_n), $u_n \in E$. $u_n(p)$ p-th term of u_n. I set

$$u_n(p) = \delta_{n,p} = \begin{cases} 1 \text{ if } n = p \\ 0 \text{ otherwise} \end{cases}$$

$$\|u_n\|_1 = \sum_{p=0}^{\infty} |u_n(p)| = |u_n(n)| = 1$$

Thus $u_n \in B_f(0,1) \forall n$.
If $v \in l^1(\mathbb{N},\mathbb{R})$

$$|v(p)| \leq \sum_{p=0}^{\infty} |v(p)| = \|v\|_1$$

if $\|v_n - v\|_1 \xrightarrow[n\to\infty]{} 0$ then $\forall p, v_n(p) \to v(p)$. Assume that $(v_n) = (u_{\phi(n)})$ is a subsequence of (u_n) that converges to v for $\|.\|$. I fix $p \in \mathbb{N}$, $v_n(p) = u_{\phi(n)}(p) \xrightarrow[n\to\infty]{} v(p)$, but $v_n(p) \xrightarrow[n\to\infty]{} 0$, so $v(p) = 0 \forall p$. v: null sequence, also

$$\|v_n\|_1 = 1 \forall n \text{ and } \|v_n\|_1 \xrightarrow[n\to\infty]{} \|v\|_1$$

contradiction □

5.3 Equivalent norms

Definition 5.13. Two norms N_1 and N_2 on E are equivalent $(N_1 \sim N_2)$ if $\exists c_1, c_2 > 0$ such that

- $N_1(u) \leq c_1 N_2(u) \quad \forall u \in E$
- $N_2(u) \leq c_2 N_1(u) \quad \forall u \in E$

$\exists c > 0$ such that

$$cN_1(u) \leq N_2(u) \leq cN_1(u)$$

Remark 5.14. If $N_1 \sim N_2$ and $N_2 \sim N_3$, then $N_1 \sim N_3$

Definition 5.15. The norms N_1 and N_2 are **topologically equivalent** if they define the same open sets.

Theorem 5.16. Let N_1, N_2 be two norms, then:

$$N_1, N_2 \text{ topologically equivalent } \Leftrightarrow N_1, N_2 \text{ equivalent}$$

Example 5.17.
1. $E = \mathcal{C}([0,1], \mathbb{R})$
2. $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$
3. $\|f\|_1 = \int_0^1 |f(x)|\, dx$
   We notice that $\|f\|_1 \leq \|f\|_\infty$. Is there $\exists c > 0$ such that

$$\|f\|_\infty \leq c\|f_1\| \forall f \in E$$

? To see this, construct a sequence (f_n) in E such that $\|f_n\|_1 \to 0$ but $\|f_n\|_\infty \not\to 0$

Theorem 5.18. Let E be a finite-dimensional space. Then all norms on E are equivalent.

Proof. Since E is finite-dimensional, there exists a basis for E and thus a linear isomorphism between E and $\mathbb{R}^n$ (or $\mathbb{C}^n$). Consequently, we can reduce the problem to the study of norms on $\mathbb{R}^n$.

Consider the norm $\|\cdot\|_1$ on E and define the associated unit sphere:

$$S = \{x \in E : \|x\|_1 = 1\}.$$

In a finite-dimensional space, the unit sphere S is compact (this relies on the fact that in $\mathbb{R}^n$, closed and bounded sets are compact).

The function

$$f : S \to \mathbb{R}, \quad f(x) = \|x\|_2$$

is continuous because $\|\cdot\|_2$ is a norm (and thus a continuous function). By Weierstrass's theorem, f attains its bounds on S. Therefore, there exist:

- A minimum $m = \min_{x\in S} f(x) > 0$ (the strict inequality $m > 0$ is explained by the fact that $x \neq 0$ for $x \in S$).
- A maximum $M = \max_{x\in S} f(x)$.

Let $x \in E$ be arbitrary, $x \neq 0$. We write $x = \|x\|_1\, y$ with $y = \frac{x}{\|x\|_1}$ which belongs to S. Then,

$$\|x\|_2 = \|x\|_1\, \|y\|_2.$$

However, since $y \in S$, we have

$$m \leq \|y\|_2 \leq M.$$

Thus,

$$m\, \|x\|_1 \leq \|x\|_2 \leq M\, \|x\|_1.$$

By setting $c = m$ and $C = M$, we obtain exactly the equivalence of norms.

For $x = 0$, the inequality is trivial because $\|0\|_1 = \|0\|_2 = 0$.

□

5.4 Complements on normed vector spaces

5.4.1 Sequences of functions

X set ($X \subset \mathbb{R}$), $f_n : X \to \mathbb{R}(\mathbb{C})$ and $(f_n)_{n\in\mathbb{N}}$. Useful for the rest of the chapter: $B(X, \mathbb{R})$ denotes a set of functions $f : X \to \mathbb{R}$ bounded

5.4.2 Simple convergence:

Definition 5.19. $(f_n)_{n\in\mathbb{N}}$ converges pointwise to f if $\forall x_0 \in X$, $f_n(x_0) \xrightarrow[n\to\infty]{} f(x_0)$ (does not come from a norm).

5.4.3 Uniform convergence:

Definition 5.20. $f \in B(X, \mathbb{R})$ if $\sup_{x\in X} |f(x)| = \|f\|_\infty < \infty$ (f bounded on X).
Uniform convergence: $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ such that $\forall n \geq N \forall x \in X |f_n(x) - f(x)| < \varepsilon$ equivalent to

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \text{ such that } \forall n \geq N, \|f_n - f\|_\infty < \varepsilon$$

$f_n \to f$ in $(B(X, \mathbb{R}), \|\cdot\|_\infty)$

Definition 5.21. Uniform limit of continuous functions: $X = [a, b]$, $\mathcal{C}([a, b], \mathbb{R}) \subset B([a, b], \mathbb{R})$(vector subspaces). $\mathcal{C}([a, b], \mathbb{R})$ is closed in $(B([a, b], \mathbb{R}), \|\cdot\|_\infty)$

5.4.4 Series with values in a normed vector space.

Definition 5.22. Let $(E, \|\cdot\|_\infty)$ n.v.s[a], $(u_n)_{n\in N}$ sequence in E. The series $\sum u_n$ converges in $(E, \|\cdot\|)$ if the sequence $S_N = \sum_{n=0}^{N} u_n$ converges in $(E, \|\cdot\|)$. $\lim_{N\to\infty} S_N$ denoted $\sum_{n=0}^{\infty} u_n (\in E)$

[a] normed vector space

Remark 5.23. If $\sum u_n$ and $\sum v_n$ converge, then

- $\sum u_n + v_n$ converges and $\sum \lambda u_n$ converges
- $\sum_{n=0}^{\infty} u_n + v_n = \sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} v_n$
- $\sum_{n=0}^{\infty} \lambda u_n = \lambda \sum_{n=0}^{\infty} u_n$

5.4.5 Normal convergence

Definition 5.24. $\sum u_n$ converges normally in $(E, \|\cdot\|)$ if $\sum \|u_n\|$ converges in $\mathbb{R}$.

Example 5.25. $E = \mathbb{R}$, $\|x\| = |x|$. normal conv. $=$ absolute conv. ($\sum u_n$ converges)

Example 5.26. $\sum u_n$ can converge without converging normally, like: $u_n = \frac{(-1)^n}{n}$

Theorem 5.27. If $(E, \|\cdot\|)$ is complete, every normally convergent series is convergent and

$$\|\sum_{n=0}^{\infty} u_n\| \leq \sum_{n=0}^{\infty} \|u_n\|$$

Proof. $S_n = \sum_{k=0}^n u_k$ and $T_n = \sum_{k=0}^n \|u_k\|$

$$n > p \quad \|S_n - S_p\| = \|\sum_{k=p+1}^{n} u_k\| \leq \sum_{k=p+1}^{n} \|u_k\| = T_n - T_p = |T_n - T_p|$$

(T_n) converges in $\mathbb{R}$, therefore (T_n) is Cauchy:

$$\forall \varepsilon > 0, \exists N \text{ such that } \forall n > p \geq N |T_n - T_p| \leq \varepsilon$$

therefore (S_n) is Cauchy in $(E, \|\cdot\|)$. E complete: (S_n) converges to $S \in E$. □

5.5 Continuous Linear Maps

For any section B_E denotes a ball closed !

Let E, F 2 normed vector spaces with $\|\cdot\|_E$ and $\|\cdot\|_F$ the associated norms,

- $A \in \mathcal{L}(E, F)$
- $\lambda A \in \mathcal{L}(E, F)$ and $\lambda Ax = \lambda(Ax)$
- $A + B \in \mathcal{L}(E, F)$ and $(A + B)x = Ax + Bx$
- $0x = 0_F \ \forall x \in E$

$$\mathcal{L}(E) = \mathcal{L}(E, E)$$

- $(AB)x = A(Bx)$ where $AB = A \circ B$
- $(\lambda A)B = \lambda(AB)$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $0A = 0$
- $AB \neq BA$ (in general)
- $A(BC) = (AB)C$

Theorem 5.28. Let $A \in \mathcal{L}(E, F)$. The following properties are equivalent:

1. $A : E \to F$ is continuous
2. A is continuous at 0_E
3. $\exists C \geq 0$ such that
$$\|Ax\|_F \leq C\|x\|_E \quad \forall x \in E$$
this is called that A is bounded
4. A is bounded on $B_E(0, R)$ $\forall R > 0$

We say that A is bounded (if A is continuous and linear)

Proof.
- 1) $\Rightarrow$ 2) : obvious
- 2) $\Rightarrow$ 3) :
  - Hyp: $\forall \varepsilon > 0, \exists \delta > 0$ tq $\|x - 0_E\|_E \leq \delta \Rightarrow \|Ax - A0_E\|_F \leq \varepsilon$ $\|x\|_E \leq \delta \Rightarrow \|Ax\|_F \leq \varepsilon$
  - $\varepsilon = 1 \exists \delta > 0$ tq $\|x\|_E \leq \delta \Rightarrow \|Ax\|_F \leq 1$
  - Let $x \in E$ and $x \neq 0_E$

- $y = \frac{\delta}{\|x\|_E} x$ therefore $\|y\|_E = \delta \Rightarrow \|Ay\|_F \leq 1$
- $Ay = \frac{\delta}{\|x\|_E} Ax$ and A linear
- $\|Ay\|_F = \frac{\delta}{\|x\|_E} \|Ax\|_F \leq 1 \Rightarrow \|Ax\|_F \leq \frac{1}{\delta} \|x\|_E$

- 3) $\Rightarrow$ 1)
  - I fix $x_0 \in E$. to see: A continuous at x_0?
  - $\|Ax - Ax_0\|_F = \|A(x - x_0)\|_F \leq C\|x - x_0\|_E$
  - Therefore if $\|x - x_0\|_E \leq \frac{\varepsilon}{c} = \delta(\varepsilon)$, $\|Ax - Ax_0\|_F \leq \varepsilon$

□

Notation.

$$B(E, F) = \{A \in \mathcal{L}(E, F) : A \text{ continuous } \}$$
$$B(E, E) = B(E)$$

Lemma 5.29. If E is of finite dimension, then

$$\mathcal{L}(E, F) = B(E, F)$$

It is false if $\dim E = \infty$

Proof. $(e_1, \ldots, e_n)$ basis of E. On E all norms are equivalent.

- $\|x\|_E$ given norm.
- $\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$

where $x = \sum_{i=1}^n x_i e_i$

$$\|Ax\|_F = \|\sum_{i=1}^n x_i Ae_i\| = \sum_{i=1}^n |x_i| \|Ae_i\|_F$$

$$\|Ax\|_F \leq \|x\|_\infty \times \sum_{i=1}^n \|Ae_i\|_F = C\|x\|_\infty$$

$(\|x\|_\infty\| \leq C'\|x\|_E)$. Therefore: $\|Ax\|_F \leq CC'\|x\|_E$. So: $A \in B(E, F)$ □

5.5.1 Norm on $B(E, F)$

Theorem 5.30. Let $A \in B(E, F)$, we set $\|A\| = \sup_{x \in E, \|x\|_E \leq 1} \|Ax\|_F = \sup_{x \in B_E(0,1)} \|Ax\|_F$

1. $\| \cdot \|$ is a norm on $B(E, F)$ called the uniform norm.
2. We have: $\|Ax\|_F \leq \|A\| \|x\|_E \quad \forall x \in E$
3. $\|A\| =$ the smallest constant C such that $\|Ax\|_F \leq C\|x\|_E \quad \forall x \in E$

Remark 5.31.

1. We can write $\|A\|_{B(E,F)}$ instead of $\|A\|$
2. Sometimes we find $|||A|||$ for $\|A\|$
3. Let $I^+ =$ the set of $C \geq 0$ such that $\|Ax\|_F \leq C\|x\|_E \quad \forall x \in E$.
   $I^+ \neq \emptyset$ (because $A \in B(E, F)$) and $I^+ \subset [0, +\infty[$. (2) and (3) say that $\|A\|$ is the smallest element

of I^+
$$\inf I^+ = \min I^+ = \|A\|$$

Proof. 1. $A \in B(E,F) \Leftrightarrow \sup_{x\in B_E(0,1)} \|Ax\|_F < \infty \Leftrightarrow \|A\|$ well defined.

$$\|(A+B)x\|_F = \|Ax+Bx\|_F \leq \|Ax\|_F + \|Bx\|_F$$
$$\Rightarrow \sup_{x\in B_E(0,1)} \|(A+B)x\|_F \leq \sup_{x\in B_E(0,1)} \|Ax\|_F + \sup_{x\in B_E(0,1)} \|Bx\|_F$$

$$\|A+B\| \leq \|A\| + \|B\| \text{ and } A,B \in B(E,F) \Rightarrow A+B \text{ also}$$
$$\|\lambda A\| = |\lambda|\|A\| \text{ and } A \in B(E,F) \Rightarrow \lambda A \text{ also}$$

Si $\|A\| = 0$, alors $\|Ax\|_F = 0 \forall x \in B_E(0,1) \Rightarrow Ax = 0_F \forall x \in B_E(0,1)$

$$Ax = \|x\|_E A\frac{x}{\|x\|_E}$$

$$Ax = 0_F \forall x \in E \Rightarrow A = 0_{L(E,F)}$$
$$C \in I^+ \text{ if } \|Ax\|_F \leq C\|x\|_E \quad \forall x \in E$$
$$\|A\| \in I^+ \Rightarrow \|Ax\|_F \leq \|A\|\|x\|_E \forall x$$

- Clear if $x = 0_E$.
- Si $x \neq 0_E$, $y = \frac{x}{\|x\|_E} \in B_E(0,1)$ donc

$$\|Ay\|_F = \frac{1}{\|x\|_E}\|Ax\|_F \leq \|A\| \Rightarrow \|Ax\|_F \leq \|A\|\|x\|_E$$

Soit $C \in I^+$ donc
$$\|Ax\|_F \leq C\|x\|_E$$
donc $\|Ax\|_F \leq C \quad \forall x \in B_E(0,1)$, donc $\|A\| \leq C$, alors

$$\|A\| = \min I^+ = \text{"best constant } C\text{"}$$

□

Example 5.32. $E = \mathcal{C}([a,b],\mathbb{R})$, $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$, $F = \mathbb{R}$, $u \in \mathcal{C}([a,b],\mathbb{R})$

$$A : E \longrightarrow F$$
$$f \longmapsto A(f) = \int_a^b f(x)u(x)\,dx.$$

<u>A is bounded</u>: to see: $\exists C \geq 0$ such that

$$\left|\int_a^b f(x)u(x)\,dx\right| \leq C \sup_{x\in[a,b]} |f(x)|$$

?

$$\left|\int_a^b f(x)u(x)\,dx\right| \leq \int_a^b |f(x)||u(x)|\,dx \leq \int_a^b \|f\|_\infty |u(x)|\,dx = \|f\|_\infty \int_a^b |u(x)|\,dx$$

$$C = \int_a^b |u(x)|\,dx \text{ suitable}$$

(In fact $\|A\| = \int_a^b |u(x)|\,dx$). $E = \mathcal{C}^1([0,1],\mathbb{R})$ equipped with $\|f\|_\infty$, $F = \mathbb{R}$, $Af = f'(0)$ linear but not continuous. We construct a sequence (f_n) in E such that $\|f_n\|_E \xrightarrow[n\to\infty]{} 0$ but $\|Af_n\|_F \not\to 0$

$$f_n(x) = \frac{1}{n}\sin(nx)$$

Proposition 5.33. Let $A \in B(E,F)$ and $\|A\| = \sup_{\|x\|_E \leq 1} \|A\|_F$ a uniform norm. $\|A\|$ = smallest c such that

$$\|Ax\|_F \leq c\|x\|_E \quad \forall x \in E$$

Proof. $E = \mathcal{C}([a,b],\mathbb{R})$ and $\|f\|_1 = \int_a^b |f(x)|\,dx$ norm on $\mathcal{C}([a,b],\mathbb{R})$. I fix $m \in \mathcal{C}([a,b],\mathbb{R})$ and $A : f \to mf$. $Af(x) = m(x)f(x)$.

- $A \in L(E)$ obvious
- $A \in B(E)$?

Find $c \geq 0$ such that

$$\|Af\|_1 \leq c\|f\|_1 \quad \forall f \in E$$

$$\|Af\|_1 = \int_a^b |m(x)f(x)|\,dx$$

$$|m(x)f(x)| \leq |m(x)||f(x)| \leq \|m\|_\infty |f(x)|$$

$$\|m\|_\infty = \sup_{x\in[a,b]} |m(x)|$$

$$\int_a^b |m(x)f(x)|\,dx \leq \|m\|_\infty \int_a^b |f(x)|\,dx = \|m\|_\infty \|f\|_1$$

$$c = \|m\|_\infty$$

We have: $A \in B(E)$ and $\|A\| \leq \|m\|_\infty$. Let's show that $\|A\| = \|m\|_\infty$

$$\|A\| = \sup_{\|f\|_1 \leq 1} \|Af\|_1 \overset{?}{=} \|m\|_\infty = \sup I \text{ with } I = \{\|Af\|_1 : \|f\|_1 \leq 1\}$$

Let's denote $\alpha = \sup I$

1. α upper bound of I
2. $\exists (a_n)\, a_n \in I$ with $a_n \xrightarrow[n\to\infty]{} \alpha$

In our case:

- Goal: find a sequence $f_n \in E$ $\|f_n\|_1 \leq 1$ and $\|Af_n\|_1 \to \|m\|_\infty$

$a_n = \|Af_n\|_1$ $\|m\|_\infty$ = sup of the function $|m|$ on $[a,b]$.

- $|m|$ continuous: $\exists x_0 \in [a,b]$ such that $\|m\|_\infty = |m|(x_0)$

$$|m|(x) = |m(x)|$$

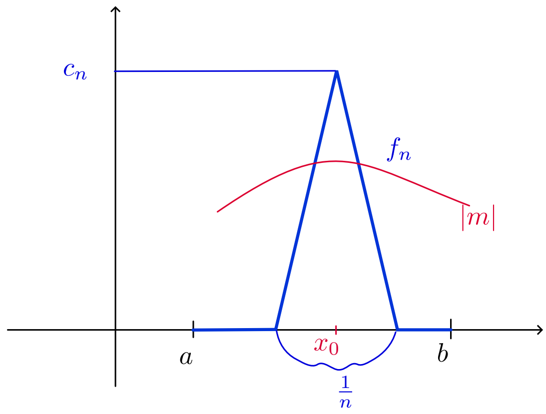

Figure 5.2: f_n

$$|m(x)f_n(x)| = |Af(x)| \text{ close to } |m(x_0)||f_n(x)|$$

$\|f_n\|_1 = 1$ if $c_n \leq 2n$

$$f_n(x) = \begin{cases} 0 \text{ if } a \leq x \leq x_0 - \frac{1}{2n} \\ 2n(1 - n|x - x_0|) \text{ if } |x - x_0| \leq \frac{1}{2n} \\ 0 \text{ if } x_0 + \frac{1}{2n} \leq x \leq b \end{cases}$$

Figure 5.3: f_n

$$|m(x)f_n(x) - m(x_0)f_n(x)| \leq |m(x) - m(x_0)||f_n(x)| \leq \varepsilon_n |f_n(x)|$$

Where $f_n(x) \neq 0$ $|x - x_0| \leq \frac{1}{n}$ so

$$|m(x) - m(x_0)| \leq \varepsilon_n \quad \varepsilon_n \xrightarrow[n\to\infty]{} 0$$

then m continuous at x_0.

$$\|Af_n\|_1 = \int_a^b |m(x)f_n(x)|\,dx \leq \int_a^b |m(x) - m(x_0)||f_n(x)|\,dx + \int_a^b |m(x_0)||f_n(x)|\,dx$$

- 1^{st} term: $\leq \varepsilon_n \|f_n\|_1 = \varepsilon_n$
- 2^{nd} term: $:= \|m\|_\infty \|f_n\|_1 = \|m\|_\infty$

Then:

$$\begin{aligned} &\|f_n\|_1 = 1 \\ &\|Af_n\|_1 \to \|m\|_\infty \\ &\text{so } \|A\| = \|m\|_\infty \end{aligned}$$

□

Proposition 5.34. The case of $B(E)$:
If $A, B \in B(E)$, $A \circ B$ (denoted AB) $\in B(E)$ and

$$\|AB\| \leq \|A\|\|B\|$$

(very useful)

Proof.

$$\|A\underbrace{Bx}_{Y}\|_E \leq \|A\|\|Bx\|_E \leq \overbrace{\|A\|\|B\|}^{c}\cdot\|x\|_E$$

thus $\|AB\| \leq \|A\|\|B\|$ □

Theorem 5.35. If N_1, N_2 are two norms on E. N_1 and N_2 are topologically equivalent $\Leftrightarrow$ N_1 and N_2 are equivalent.

Proof. E_1 is (E, N_1), $E_2 = (E, N_2)$.
N_1 and N_2 topologically equivalent means exactly:

1. $Id : E_1 \to E_2$ are continuous
2. and $Id : E_2 \to E_1$

Therefore:

1. Ω open for $N_2 \Rightarrow \Omega$ open for N_1
2. Ω open for $N_1 \Rightarrow \Omega$ open for N_2

1. $\Leftrightarrow N_2(Id\,u)(= N_2(u)) \leq c_1 N_1(u)$
2. $\Leftrightarrow N_1(Id\,u)(= N_1(u)) \leq c_2 N_2(u)$

because Id continuous and linear, therefore bounded $\exists c$ such that $\underbrace{Id\,u}_{\in E_2} \leq c \underbrace{u}_{E_1}$ therefore $N_2(Id\,u) \leq cN_1(u)$

(1) and (2) $\Leftrightarrow$ N_1 and N_2 equivalent. □

5.6 The norm of matrices

$A \in \mathcal{M}_n(\mathbb{C})$ identified with $A \in L(\mathbb{C}^n)$

$$\left((Ax)_i = \sum_{j=1}^{n} a_{i,j}x_j\right) \qquad x = (x_1, \ldots, x_n) \in \mathbb{C}^n$$

- $(x|y) = \sum_{i=1}^n \overline{x_i} y_i$
- $\|x\| = (x|x)^{\frac{1}{2}} = \left(\sum_{i=1}^n |x_i|^2\right)^{\frac{1}{2}}$

Adjoint matrix A^* $(A^*)_{i,j} = \overline{(A)_{j,i}}$

$$(x|Ay) = (A^*x|y) \quad \forall x, y$$

5.6.1 <u>Good norm</u> on $L(\mathbb{C}^n)$ (or on $\mathcal{M}_n(\mathbb{C})$)

$\|A\|$ uniform norm on $L(\mathbb{C}^n)$ $(= B(\mathbb{C}^n))$ obtained from $\|\cdot\|_2$

Lemma 5.36.

$$\|A\| = \|A^*\| = \|A^*A\|^{\frac{1}{2}}$$

Proof. $\|x\|_2 = \sup_{\|y\|_2 \leq 1} |(y|x)|$. Therefore:

$$\|A\| = \sup_{\|x\|_2 \leq 1} \|Ax\|_2 = \sup_{\|x\|_2 \leq 1, \|y\|_2 \leq 1} |(y|Ax)|$$

$$(y|Ax) = (A^* y|x) = \overline{(x|A^* y)}$$

therefore $|(y|Ax)| = |(x|A^* y)|$

$$\|A\| = \sup_{\|x\| \leq 1, \|y\| \leq 1} |(x|A^* y)| = \|A^*\|$$

$$\|A^* A\| \leq \|A^*\| \|A\| = \|A\|^2 = \sup_{\|x\| \leq 1} \|Ax\|^2$$

$$\begin{aligned} \|Ax\|^2 = (Ax|Ax) = (x|A^* Ax) &\leq \|x\| \|A^* Ax\| \text{ (Cauchy-Schwarz)} \\ &\leq \|x\| \|A^* A\| \|x\| = \|A^* A\| \|x\|^2 \end{aligned}$$

$$\|Ax\|^2 \leq \|A^* A\| \|x\|^2$$

$$\|Ax\|_2 \leq \|A^* A\|^{\frac{1}{2}} \|x\|_2 \Rightarrow \|A\|^2 \leq \|A^* A\|^{\frac{1}{2}}$$

$$\|A\| = \|A^* A\|^{\frac{1}{2}}$$

□

5.6.2 How to "calculate" $\|A\|$?

Theorem 5.37. $\|A\| = \max_{1 \leq i \leq n} \mu_i$ with $\mu_i = \lambda_i^{\frac{1}{2}}$ where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}^+$ eigenvalues of $A^* A$.

Proof.

$$\|A\| = \|A^* A\|^{\frac{1}{2}}$$

To show: $\|A^* A\| = \max_{1 \leq i \leq n} \lambda_i$ $(\lambda_i \geq 0)$

$$(AB)^* = B^* A^*$$

$$(A^* A)^* = A^* A^{**} = A^* A$$

Let $B = A^* A$, $B = B^*$ and $(x|Bx) = (x|A^* Ax) = (Ax|Ax) = \|Ax\|^2 \geq 0$. Therefore:

$$\forall x, \quad (x|Bx) \geq 0$$

There exists an o.n.b. $(u_1, \ldots, u_n)$ of $\mathbb{C}^n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that

$$Bu_i = \lambda_i u_i \quad 1 \leq i \leq n$$

$$\lambda_i = (u_i|\lambda_i u_i) = (u_i|Bu_i) \geq 0$$

If $u = \sum_{i=1}^n x_i u_i$ $\|u\|^2 = \sum_{i=1}^n |x_i|^2$

$$Bu = \sum_{i=1}^n x_i B u_i = \sum_{i=1}^n \lambda_i x_i u_i$$

$$\|Bu\|^2 = \sum_{i=1}^n \lambda_i^2 |x_i|^2 \leq \max \lambda_i^2 \cdot \sum_{i=1}^n |x_i|^2 = \max \lambda_i^2 \|u\|^2$$

$$\|B\| \leq \max_{1 \leq i \leq n} \lambda_i$$

If $\lambda_1 = \max_{1 \leq i \leq n} \lambda_i$

$$\|Be_1\| = \|\lambda_1 e_1\| = \lambda_1 \|e_1\| \leq \|B\| \|e_1\|$$

therefore $\|B\| \geq \lambda_1$ □

5.6.3 How to bound $\|A\|$

Proposition 5.38. We have: $\|A\| \leq \|A\|_{HS}$ where

$$\|A\|_{HS}^2 = \sum_{1 \leq i,j \leq n} |a_{i,j}|^2$$

Proof.

$$\mathcal{M}_n(\mathbb{C}) \sim \mathbb{C}^{n \times n}$$

$\| \cdot \|_{HS}$ canonical norm on $\mathbb{C}^{n \times n}$!

$$(Ax)_i = \sum_{i=1}^{n} a_{i,j} x_j$$

$$(y|Ax) = \sum_{i=1}^{n} \overline{y_i} \sum_{j=1}^{n} a_{i,j} x_j = \sum_{1 \leq i,j \leq n} a_{i,j} \overline{y_i} x_j$$

Let $b_{i,j} = y_i \overline{x_j}$

$$(y|Ax) = \sum_{i,j} \overline{b_{i,j}} a_{i,j}$$

$$|(y|Ax)| \leq \left(\sum_{i,j} |a_{i,j}|^2 \right)^{\frac{1}{2}} \times \left(\sum_{i,j} |b_{i,j}|^2 \right)^{\frac{1}{2}}$$

$$\left(\sum_{i,j} |b_{i,j}|^2 \right)^{\frac{1}{2}} = \left(\sum_{1 \leq i,j \leq n} |y_i|^2 |x_i|^2 \right)^{\frac{1}{2}} = \left(\sum_{1 \leq i \leq n} |y_i|^2 \right)^{\frac{1}{2}} \times \left(\sum_{1 \leq j \leq n} |x_j|^2 \right)^{\frac{1}{2}} = \|y\| \|x\|$$

$$|(y|Ax)| \leq \|A\|_{HS} \|x\| \|y\| \Rightarrow \|A\| \leq \|A\|_{HS}$$

□

CHAPTER 6

System of differential equations

$$(E)\begin{cases} x_1'(t) = a_{1,1}x_1(t) + \ldots + a_{1,n}x_n(t) + f_1(t) \\ \vdots \\ x_n'(t) = a_{n,1}x_1(t) + \ldots + a_{n,n}x_n(t) + f_n(t) \end{cases}$$

$$x(t) = (x_1, \ldots, x_n(t)) \text{ or } \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$A = [a_{i,j}]_{1 \le i,j \le n} \quad f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

$$\begin{aligned} f : \mathbb{R} &\longrightarrow \mathbb{C}^n \\ A &\longmapsto f(A) = \mathcal{M}_n(\mathbb{C}) \end{aligned}$$

$$x'(t) = Ax(t) + f(t)$$

$$(H)\; x'(t) = Ax(t)$$

$$(C)\begin{cases} x'(t) = Ax(t) + f(t) \\ x(0) = x_0 \in \mathbb{C}^n \end{cases}$$

Solution on I:$f : I \to \mathbb{C}^n$ with $I \subset \mathbb{R}$ interval (f assumed continuous). $x : I \to \mathbb{C}^n$ of class $\mathcal{C}^1$ such that (C) verified $\forall t \in I$

- If $n = 1$ $A = a \in \mathbb{C}$. Solution of (H): $x(t) = e^{ta}x_0$ with $x_0 \in \mathbb{C}$

$$e^a = \sum_{n=0}^{\infty} \frac{a^n}{n!}$$

But:define e^A

Theorem 6.1. Let $A \in \mathcal{M}_n(\mathbb{C})$ ($A \in B(E)$ where $(E, \|\cdot\|)$ complete!)

1. The series $\sum_{n\in\mathbb{N}} \frac{A^n}{n!}$ converges in $(\mathcal{M}_n(\mathbb{C}), \|\cdot\|)$, its sum $\sum_{n=0}^{\infty} \frac{A^n}{n!}$ denoted e^A is called the exponential of A.
2. $\|e^A\| \le e^{\|A\|}$
3. $\|e^A - \sum_{n=0}^{N} \frac{A^n}{n!}\| \le \frac{\|A\|^{N+1}}{(N+1)!} e^{\|A\|} (\le \frac{\|A\|_{HS}^{N+1}}{(N+1)!} e^{\|A\|_{HS}})$
4. $e^A e^B = e^B e^A$ if $AB = BA$

5. $Be^A = e^A B$ if $AB = BA$

Proof. -

1. $\|AB\| \leq \|A\| \|B\|$ (because $\|\cdot\|$ uniform norm!) therefore $\|A^n\| \leq \|A\|^n$. $\sum_{n\in\mathbb{N}} \frac{\|A\|^n}{n!}$ (numerical series!) converges to $e^{\|A\|}$ therefore $\sum_{n\in\mathbb{N}} \frac{A^n}{n!}$ converges normally in $\mathcal{M}_n(\mathbb{C})$.

   $\mathcal{M}_n(\mathbb{C})$ is complete (like $B(E)$ if E is complete) therefore $\sum_{n\in\mathbb{N}} \frac{A^n}{n!}$ converges in $\mathcal{M}_n(\mathbb{C})$.

2. OK also

3. $$\|e^A - \sum_{n=0}^{N} \frac{A^n}{n!}\| = \| \sum_{n=N+1}^{\infty} \frac{A^n}{n!} \| \leq \sum_{n=N+1}^{\infty} \frac{\|A\|^n}{n!}$$

   We denote $f(x) = e^x$

   $$f(x) = \sum_{n=0}^{N} f^{(n)}(0)\frac{x^n}{n!} + \frac{x^{N+1}}{(N+1)!} f^{(N+1)}(y) \quad (y \text{ between } 0 \text{ and } x)$$

   $x = \|A\|$

4. $$e^A e^B = e^{A+B} \text{ if } AB = BA$$

   $$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2 \quad (\text{ if } AB = BA)$$

   $$(A+B)^n = \sum_{p=0}^{n} C_n^p A^{n-p} B^p$$

   Same proof if $A = a, B = b$ with $a, b \in \mathbb{R}$

5. exercise

□

Remark 6.2.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad BA = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$A^2 = 0 \quad e^A = \mathcal{I} + A \quad e^B = \mathcal{I} + B \text{ where } \mathcal{I} \text{ identity}$$

$$e^A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad e^B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad e^A e^B = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$e^{A+B} =? \quad A + B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (A+B)^2 = \mathcal{I}$$

$$C = A + B \quad C^{2p} = \mathcal{I} \quad C^{2p+1} = C$$

$$e^C = \sum_{p=0}^{\infty} \frac{C^{2p}}{2p!} + \sum_{p=0}^{\infty} \frac{C^{2p+1}}{(2p+1)!} = \mathcal{I} \underbrace{\sum_{p=0}^{\infty} \frac{1}{2p!}}_{=\text{ch}(1)} + C \underbrace{\sum_{p=0}^{\infty} \frac{1}{(2p+1)!}}_{=\text{sh}(1)}$$

$$e^{A+B} = \text{ch}(1)\mathcal{I} + \text{sh}(1)C = \begin{pmatrix} \text{ch}(1) & \text{sh}(1) \\ \text{sh}(1) & \text{ch}(1) \end{pmatrix} \neq e^A \times e^B$$

Proposition 6.3. $A \in \mathcal{M}_n(\mathbb{C})$ (or $B(E)$)

1. $e^{tA} e^{sA} = e^{(s+t)A} \qquad s, t \in \mathbb{R}$
2. $(e^{tA})^{-1} = e^{-tA} \qquad t \in \mathbb{R}$

3. $\frac{d}{dt}(e^{tA}) = Ae^{tA} \qquad t \in \mathbb{R}$

Proof. -

1. OK because tA commutes with sA
2. $e^{tA}e^{-tA} = e^{-tA}e^{tA} = e^{0A} = \mathcal{I}$ therefore $(e^{tA})^{-1} = e^{-tA}$
3. To calculate: $\lim_{\varepsilon \to 0} \frac{e^{(t+\varepsilon)A} - e^{tA}}{\varepsilon} = ?$

$$e^{tA}(\frac{e^{\varepsilon A} - \mathcal{I}}{\varepsilon})$$

$$\begin{aligned} e^{\varepsilon A} - \mathcal{I} &= \sum_{n=1}^{\infty} \frac{(\varepsilon A)^n}{n!} \quad n = 1 + p \\ &= \varepsilon A \times \sum_{p=0}^{\infty} \frac{(\varepsilon A)^p}{(p+1)!} \\ &= \varepsilon A \left(\mathcal{I} + \| \sum_{p=1}^{\infty} \frac{(\varepsilon A)^p}{(p+1)!} \| \right) \\ &= \varepsilon A \left(\mathcal{I} + R(\varepsilon) \right) \end{aligned}$$

$$\frac{e^{\varepsilon A} - \mathcal{I}}{\varepsilon} = A + AR(\varepsilon)$$

to see: $\|AR(\varepsilon)\| \xrightarrow[\varepsilon \to 0]{} 0$ then $\lim_{\varepsilon \to 0} \frac{e^{\varepsilon A} - \mathcal{I}}{\varepsilon} = A$

$$\|AR(\varepsilon)\| \leq c\varepsilon \xrightarrow[\varepsilon \to 0]{} 0$$

$$\|R(\varepsilon)\| \leq \sum_{p=1}^{\infty} \frac{\varepsilon^p \|A\|^p}{(p+1)!} = \varepsilon \sum_{p=1}^{\infty} \frac{\varepsilon^{p-1} \|A\|^p}{(p+1)!} \leq \varepsilon e^{\|A\|}$$

If $|\varepsilon| \leq 1$

$$\frac{\varepsilon^{p-1} \|A\|^p}{(p+1)!} \leq \frac{\|A\|^p}{p!}$$

□

6.1 Application to DE system

$$(H)\; x'(t) = Ax(t)$$

Theorem 6.4. The set $\mathrm{Sol}(H)$ of solutions to (H) is given by

$$x(t) = e^{tA}x_0 \quad x_0 \in \mathbb{C}^n$$

Let $x(t)$ be a solution

$$\begin{aligned} y(t) &= e^{-tA}x(t) \\ \Rightarrow y'(t) &= -Ae^{-tA}x(t) + e^{-tA}x'(t) \\ &= -Ae^{-tA}x(t) + e^{-tA}Ax(t) \\ &= 0 \\ \Rightarrow y(t) &= x_0 \Rightarrow x(t) = e^{tA} \end{aligned}$$

$$(E) \quad x'(t) = Ax(t) + f(t)$$

$$x(t) = e^{tA}x_0 + \int_0^t e^{(t-s)A} f(s)\, ds$$

$$x(0) = x_0$$

Proof. Seek $x(t)$ in the form

$$x(t) = e^{tA}y(t)$$

I find that $y'(t) = e^{-tA}f(t)$

$$y(t) = x_0 + \int_0^t e^{-sA} f(s)\, ds$$

□

Bibliography

[1] Christian Gérard. *Analyse et Géométrie (OLMA251)*. fre.

[2] Christian Gérard. *Cours Magistral d'Analyse et Géométrie (OLMA251) à l'Université Paris-Saclay*. 2024-2025.

[3] Yehor Korotenko. *sci-trans-git*. Version 0.2.0-alpha. 2025. DOI: 10.5281/zenodo.15775111. URL: https://github.com/DobbiKov/sci-trans-git.